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Marcellino Gaudenzi, Matteo Madotto, Fabio Zanolin

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Marcellino Gaudenzi*

Dipartimento di Scienze Economiche e Statistiche,
Universitá di Udine, via Tomadini 30/a, 33100 Udine, Italy

Matteo Madotto

Scuola Superiore,
Universitá di Udine, via Tomadini 3/a,33100 Udine, Italy

Fabio Zanolin

Università di Udine, Dipartimento di Matematica e Informatica,
via delle Scienze 206, 33100 Udine, Italy

Abstract

Goodwin's celebrated growth cycle model has been widely studied since its introduction in 1967. In recent years several contributions have appeared with the aim of amending the original model so as to improve its economic coherence and enrich its structure. In this article we propose a new and generalized approach, within the theory of planar Hamiltonian systems, for the modeling of Goodwin-type cycles. This new approach, which includes and improves various attempts by the recent literature, is very general and fulfills the essential requirement that the orbits lie entirely in the economically feasible interval. We provide a necessary and sufficient condition for all solutions to be cycles lying entirely in the unit box. In addition, we study the period length of the cycles near the equilibrium and close to the boundary of the domain. Finally, we discuss an example of how small perturbations in the model may affect the qualitative behavior of the solutions.

Keywords: Goodwin model, nonlinear systems, growth cycle, period, stability.
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*Corresponding author

Email addresses: gaudenzi@uniud.it (Marcellino Gaudenzi), madotto.matteo@spes.uniud.it (Matteo Madotto), fabio.zanolin@uniud.it (Fabio Zanolin)

1. Introduction

Goodwin's celebrated growth cycle model (also known as Goodwin's class struggle model), which was first proposed in 1967 ([11]) and subsequently restated in 1972 ([12]), is a schematized, yet very elegant, dynamic formalization of Marx's theory of distributive conflict and an interesting example of how nonlinear dynamical systems can be used to model important economic phenomena like growth cycles. Using a linear investment function and a linear real wage bargaining function, Goodwin obtains two nonlinear differential equations of the Lotka-Volterra type in the state variables " wage share in national income" and "proportion of labor force employed".

Since its first formulation, the model has proved to be a useful framework for combining economic growth and endogenous fluctuations in a simple nonlinear model. Many authors have extended it in many different directions, especially during the 1970s and 1980s, trying to generalize the model by adopting less stringent hypotheses and introducing new economic phenomena. Some of these contributions include Desai [4], Medio [17], Desai-Shah [5], van der Ploeg [20], Di Matteo [7], Glombowski-Krüger [10], Sato [19], Mehrling [18], Asada [1] and Chiarella [3]. A more complete and recent survey of the literature can be found in Veneziani-Mohun [21].

In more recent years some authors have focused on developing models which, preserving Goodwin's basic idea of a conflicting but at the same time symbiotic interaction between capitalists and workers, could overcome one of the major shortcomings of the original model. In fact, the state variables of the model (the wage share in national income and the employment proportion) cannot by definition exceed unity, while the Lotka-Volterra equations obtained by Goodwin generate cycles which lie in the entire first quadrant of the plane. This issue was rarely noticed by the earlier literature. However, it is essential to address it in order to obtain a more realistic model and a coherent theoretical framework for the modeling of economic cycles in the two state variables considered by Goodwin. Some exceptions are Blatt [2], who suggests the use of a floor level for net investment, and Flaschel (see for instance [8]), who proposes the inclusion of additional elements such as money and fiscal policies by a state sector.

In the last decade relevant contributions to the solution of this issue have been provided by Desai *et al* [6] and Harvie *et al* [14]. The novelty of these approaches consists in the idea of properly modifying the differential equations of the original

Goodwin model, without the need of introducing economic phenomena different from those considered in the basic model and without relying on ceiling or floor mechanisms, whose economic interpretation is not always convincing.

Desai *et al* [6] present a reformulation of the Goodwin model where the real wage bargaining function has a nonlinear form (as did Phillips originally for nominal wages) and where Goodwin's restrictive assumption that all profits are always reinvested is relaxed. Harvie *et al* [14] propose a system of differential equations, inspired by mathematical models used in biology, in which each state variable has both a positive and a negative feedback effect on its own growth rate, allowing the modeling of several economic features.

In this article we propose a new and general framework for the modeling of the dynamic evolution of wage share and employment proportion, where the solution trajectories, under certain conditions, are closed orbits which never stray outside the economically feasible interval. This new approach is very general, its economic interpretation is rich, and it includes the recent attempts made by the literature as special cases.

In our generalized framework we obtain a necessary and sufficient condition for all solutions to be cycles lying entirely in the unit box. We show that the models by Desai *et al* and Harvie *et al* can be viewed as special cases of our generalized approach and that the assumptions made in these two papers are not sufficient to guarantee that the solution trajectories are closed orbits contained in the unit box as claimed. Then, we focus on the analysis of the period of the cycles in a neighborhood of the fixed point and near the frontier of the unit box. On the one hand we prove that the period of the cycles converges to the period of the corresponding linearized system as the starting point tends to the fixed point. On the other hand we prove that the period tends to ∞ as the distance of the starting point from the boundary goes to zero. The former result is known only for special cases like the basic Goodwin model, while the latter seems completely new.

Our results can also help to explain some of the empirical evidence on the Goodwin model. In fact, as shown by Harvie [13], there is evidence of a three-quarter cycle in the state variables wage share and employment proportion in many OECD countries. Our analysis shows that the period length in the missing quadrant grows to infinity as we approach the boundary, while this is not guaranteed in the other three quadrants.

Finally, we consider a slightly modified version of our generalized model in order to take inflation into account (see for example Desai [4]; van der Ploeg [20]; Flaschel [9]). We prove that in this case the equilibrium point ceases to be a center and becomes an asymptotically stable focus or node. This shows that our original system (like the basic Goodwin model) is structurally unstable. Our last result provides the condition to determine whether the equilibrium point is a focus or a node.

2. The basic Goodwin model and some recent contributions

The basic Goodwin model ([11], [12]) is deliberately schematized. Its aim is to describe the conflicting but at the same time symbiotic relationship between capitalists and workers in a purely capitalist economy, adopting a framework which is as simple as possible.

The state variables of the model are u, which represents the wage share in national income, and v, the proportion of labor force employed. Goodwin assumes a constant capital-output ratio σ, a constant exogenous labor productivity growth rate α and a constant labor force growth rate β. Moreover, he hypothesizes that the real wage bargaining function, which describes the growth rate of the real wage, is of the form $-\gamma + \rho v$, where γ and ρ are positive parameters, and that capitalists reinvest all their profits while workers do not save. From these assumptions Goodwin obtains the following system

$$\frac{\dot{u}}{u} = -(\gamma + \alpha) + \rho v. \tag{2.1}$$

$$\frac{\dot{v}}{v} = \frac{1 - u}{\sigma} - (\alpha + \beta), \tag{2.2}$$

where the dot indicates the time derivative.

Equations (2.1) and (2.2), which are the key result of Goodwin's model, form a two-dimensional autonomous system in u and v. This dynamical system is a particular case of the well-known Lotka-Volterra predator-prey model (see Lotka [16] and Volterra [22], [23]), which is used in mathematical biology to model the dynamic interaction between two populations. The system admits two equilibrium points: $(0, 0)$, which is a saddle, and $(1 - \sigma(\alpha + \beta), (\gamma + \alpha)/\rho)$, where it is assumed $\sigma(\alpha + \beta) < 1$. This second point is a center and all solution trajectories starting inside $\mathbb{R}^+ \times \mathbb{R}^+$ are closed orbits surrounding it.

One of the major problems of the Goodwin model which has been addressed and analyzed by some recent contributions is the fact that the periodic solutions of the system can exceed unity. This is inconsistent with the fact that the state variables u and v represent fractions of unity. The most recent attempts to overcome this issue are provided by Desai *et al* [6] and Harvie *et al* [14].

Using a nonlinear real wage bargaining function of the form $-\gamma + \rho(1-v)^{-\delta}$, where $\delta > 0$, and a logarithmic investment function, Desai *et al* [6] obtain a system of the form

$$\frac{\dot{u}}{u} = -(\gamma + \alpha) + \rho(1-v)^{-\delta}, \tag{2.3}$$

$$\frac{\dot{v}}{v} = (-\lambda \log(1-\bar{u}) - (\alpha + \beta)) + \lambda \log(\bar{u} - u), \tag{2.4}$$

where α, β, γ and ρ have the same meaning as above, λ is a positive parameter and $\bar{u} \in (0,1)$ is the maximum wage share the capitalists can tolerate which ensures them their reservation rate of profit. The authors assume $\rho < \gamma + \alpha$ and $\alpha + \beta < \lambda \log(\bar{u}/(1-\bar{u}))$ to ensure that the center lies in $(0,\bar{u}) \times (0,1)$.

The authors show that system (2.3)-(2.4) exhibits closed orbits lying entirely within the $(0,\bar{u}) \times (0,1)$ interval. In Section 4, however, we show that these assumptions are not sufficient to ensure that all closed orbits lie inside the required interval. In particular, it is necessary to impose a further condition on the primitives of the functions.

Another interesting proposal is that of Harvie *et al* [14]. They consider a system of the form

$$\frac{\dot{u}}{u} = k_1 u^{\mu_1}(1-u)^{\eta_1}(-(\alpha+\gamma) + \rho v), \tag{2.5}$$

$$\frac{\dot{v}}{v} = k_2 v^{\mu_2}(1-v)^{\eta_2}\Big(\frac{1}{\sigma} - (\alpha+\beta) - \frac{1}{\sigma}u\Big), \tag{2.6}$$

where $\rho > \alpha + \gamma$, $\sigma(\alpha+\beta) < 1$ and

$$k_1 > 0, \quad k_2 > 0, \quad \mu_1 \geq 0, \quad \mu_2 \geq 0, \quad \eta_1 > 0, \quad \eta_2 > 0.$$

This system allows the modeling of several economic features not captured by the original formulation. In fact, the two state variables, u and v, now have both a positive and a negative feedback effect on their own growth rates, while in the Goodwin model each state variable only affects the growth rate of the other.

The authors show that all the solution trajectories of system (2.5)-(2.6) lie in the unit box. But, as we show in Section 4, their restrictions on the size of the parameters are again not sufficient to guarantee the validity of this conclusion and further restrictions are needed.

3. The model

In this section we propose a generalized approach for the generation of Goodwin-type cycles fulfilling the essential requirement of lying entirely in an economically feasible interval.

Consider the following system of differential equations in the state variables u, the wage share in national income, and v, the proportion of labor force employed:

$$\dot{u} = uf(u)\psi(v), \tag{3.1}$$

$$\dot{v} = -vg(v)\phi(u), \tag{3.2}$$

where

1. $$f : (0, u_1) \to (0, +\infty), \quad g : (0, v_1) \to (0, +\infty),$$
   $$\phi : (0, u_1) \to \mathbb{R}, \quad \psi : (0, v_1) \to \mathbb{R},$$
   where $u_1, v_1 \in [0, 1]$, $f, g \in C$ and $\phi, \psi \in C^1$;

2. $$\phi'(u) > 0, \quad \forall u \in (0, u_1),$$
   $$\psi'(v) > 0, \quad \forall v \in (0, v_1);$$

3. $$\lim_{s \to 0^+} \phi(s) = L_1 < 0, \lim_{s \to 0^+} \psi(s) = L_2 < 0, \qquad L_1, L_2 \in \mathbb{R},$$
   $$\lim_{s \to {u_1}^-} \phi(s) > 0, \lim_{s \to {v_1}^-} \psi(s) > 0.$$

These assumptions are very general. The four functions which define the effects of the variables on the growth rates are defined on open intervals within the economically feasible range of values and are sufficiently regular. The functions ϕ and ψ, which define the effect of u on the growth rate of v and the effect of v on the growth rate of u, respectively, are increasing. This simply generalizes Goodwin's idea that the employment proportion (the prey in the Lotka-Volterra model) has a positive effect on the growth rate of the wage share (the predator) and that the latter variable has a negative impact on the growth rate of v. As u goes to zero, v increases at finite rates, while as u goes to u_1, v decreases at finite or infinite rates. The opposite holds for u as v goes to zero and v_1.

From assumptions 1., 2. and 3. it follows that there exists a unique value of u in the $(0, u_1)$ interval, say u^*, such that $\phi(u^*) = 0$, and a unique value of v in the $(0, v_1)$ interval, say v^*, such that $\psi(v^*) = 0$.

We define

$$A(u) = \int \frac{\phi(u)}{uf(u)}\, du,$$

and

$$B(v) = \int \frac{\psi(v)}{vg(v)}\, dv,$$

making the additional assumption that

4.

$$A(0^+) = A(u_1{}^-) = +\infty,$$
$$B(0^+) = B(v_1{}^-) = +\infty.$$

System (3.1)-(3.2), which can be equivalently written as

$$\frac{\dot{u}}{uf(u)} = \psi(v),$$
$$\frac{\dot{v}}{vg(v)} = -\phi(u),$$

possesses the first integral (the Hamiltonian)

$$H(u, v) = A(u) + B(v),$$

whose level lines are given by

$$A(u) + B(v) = c,$$

where c is a real constant. All the trajectories arising from system (3.1)-(3.2) lie on the level lines of this first integral.

The A function has a minimum when $\phi(u) = 0$, which occurs only at $u = u^*$. Similarly, B has a unique minimum at v^*, where $\psi(v^*) = 0$. It follows that $H(u, v)$ has a minimum, say c^*, at (u^*, v^*), which therefore is the fixed point of system (3.1)-(3.2).

Now consider $c > c^*$. The above assumptions guarantee that the equation $H(u^*, v) = c$ has exactly two solutions, $\underline{v}$ and $\overline{v}$, which satisfy

$$0 < \underline{v} < v^* < \overline{v} < v_1.$$

Analogously, the equation $H(u, v^*) = c$ has exactly two solutions, $\underline{u}$ and $\overline{u}$, which satisfy

$$0 < \underline{u} < u^* < \overline{u} < u_1.$$

Every couple (u, v) belonging to the level line $H(u, v) = c$ satisfies the following conditions:

$$0 < \underline{u} \leq u \leq \overline{u} < u_1, \quad 0 < \underline{v} \leq v \leq \overline{v} < v_1.$$

In fact, if $u > \overline{u}$ and $H(u, v) = c$, then $A(u) + B(v) = c$ and $A(\overline{u}) + B(v^*) = c$. But since $u, \overline{u} > u^*$ and $A(u)$ is increasing in the interval (u^*, u_1), we would have $B(v) < B(v^*)$, contradicting the fact that $B(v^*)$ is the minimum of B. It is thus easy to prove that the level line $H(u, v) = c$ is contained in the box $[\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$.
From the above discussion it is also clear that each level line $H(u, v) = c > c^*$ is the union of (four) graphs of smooth functions and thus has a finite length (indeed, its length is less than the perimeter of the box $[0, u_1] \times [0, v_1]$).

Now consider a solution of the system (3.1)-(3.2) satisfying the initial condition $u(0) = u_0 \in (0, u_1)$, $v(0) = v_0 \in (0, v_1)$ and $(u_0, v_0) \neq (u^*, v^*)$, so that $H(u_0, v_0) = c_0 > c^*$. The length of the regular curve $(u(t), v(t))$, $t \in [t_1, t_2]$ is given by

$$\int_{t_1}^{t_2} \sqrt{\dot{u}^2(t) + \dot{v}^2(t)}dt = \int_{t_1}^{t_2} \sqrt{u^2(t) f^2(u(t))\psi^2(v(t)) + v^2(t) g^2(v(t))\phi^2(u(t))}dt.$$

Since ϕ and ψ vanish only at u^* and v^*, respectively, and the level line $H(u, v) = c_0$ is a compact set, there exists $m > 0$ such that the integrand function is greater than m for all $t \geq 0$. Hence, the length of the curve $(u(t), v(t))$ is greater than $m(t_2 - t_1)$. On the other hand, as already remarked, the length of the level line of the Hamiltonian H is finite (less than $2(u_1 + v_1)$). Hence, there exist $t_1 < t_2$ such that $(u(t_1), v(t_1)) = (u(t_2), v(t_2))$. Thus, we conclude that for every choice of the initial condition (u_0, v_0), the corresponding solution $(u(t), v(t))$ is periodic.

The above results can be summarized in the following

Theorem 3.1. *Under assumptions 1., 2., 3. and 4. all solutions of system (3.1)-(3.2) are periodic and describe closed orbits strictly contained in the rectangle* $(0, u_1) \times (0, v_1)$.

To complete Theorem 3.1 with further details, we can add the following

Remark 3.1. *Every solution* $(u(t), v(t))$ *of (3.1)-(3.2) is contained in the box* $[\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$. $u(t)$ *is increasing in the first quadrant,* $[u^*, \overline{u}] \times [v^*, \overline{v}]$, *and in the second* *quadrant,* $[\underline{u}, u^*] \times [v^*, \overline{v}]$*, while it is decreasing in the last two quadrants.* $v(t)$ *is increasing in the second and third quadrants, while it is decreasing in the first and fourth quadrants. In the time interval* $[t, t+T)$*, where* T *is the period of the solution,* $u(t) - u^*$ *and* $v(t) - v^*$ *have exactly two zeros (see also Figure 1).*

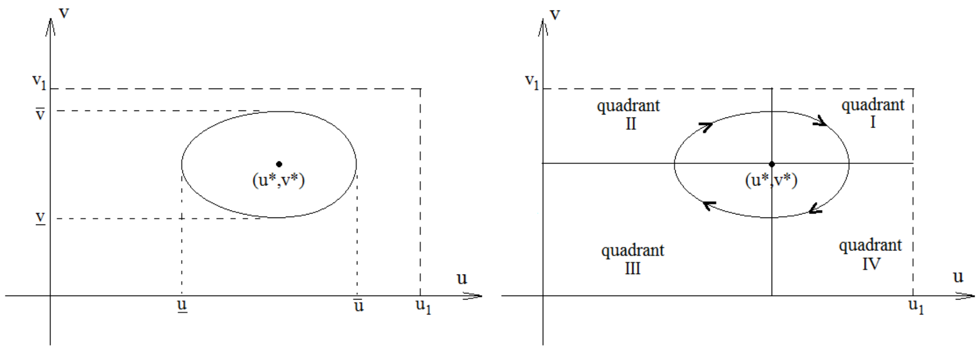

Figure 1: Goodwin-type cycles in the feasible interval.

We emphasize that Theorem 3.1 is false if any of the conditions in Assumption 4 is not fulfilled.

For example, suppose that $A(0^+) = \underline{A} < +\infty$ while the other three conditions, $A(u_1{}^-) = +\infty$, $B(0^+) = +\infty$, $B(v_1{}^-) = +\infty$, are fulfilled.

In this case, for all $c > c^*$ the points (u, v) belonging to the level line $H(u, v) = c$ satisfy the condition $0 < \underline{v} < v < \overline{v} < v_1$. The same condition, however, does not always hold for the u variable. In fact, for $c^* < c < \underline{A} + B(v^*)$ the equation $H(u, v^*) = c$ has exactly two solutions, while for $c > \underline{A} + B(v^*)$ it has only one solution.

Thus, the conclusions of Theorem 3.1 are still valid for $c^* < c < \underline{A} + B(v^*)$, but for $c > \underline{A} + B(v^*)$ there are no periodic solutions. By the previous argument on the length of the curve, we can also conclude that for $c > \underline{A} + B(v^*)$ every solution starting in the open box $(0, u_1) \times (0, v_1)$ and lying on the level line $H(u, v) = c$ will reach the boundary of the box.

In the following it will be interesting to consider the case where $A(0^+) = +\infty$, $A(u_1{}^-) = \overline{A} < +\infty$, $B(0^+) = +\infty$ and $B(v_1{}^-) = \overline{B} < +\infty$. Using the same arguments as above, we can conclude that for $c^* < c < \min\{\overline{A} + B(v^*), \overline{B} + A(u^*)\}$ the conclusions of Theorem 3.1 still hold, while for $c > \min\{\overline{A} + B(v^*), \overline{B} + A(u^*)\}$ system (3.1)-(3.2) has no periodic solutions, for every choice of the initial condition. In this latter case the solution trajectories will certainly reach either the level $u = u_1$ or the level $v = v_1$.

We can now state the following

Theorem 3.2. *Assume that hypotheses 1., 2. and 3. are fulfilled and that at least one of the four limits in Assumption 4. is finite. Then, there exists a closed curve* γ *contained in* $[0, u_1] \times [0, v_1]$ *such that*

1. *for all starting points lying in the interior of the curve* γ*, the solutions of system (3.1)-(3.2) are periodic and describe closed orbits contained in the interior of* γ*;*

2. *for all starting points lying in* $(0, u_1) \times (0, v_1)$ *but not on* γ *or in its interior, the solutions of system (3.1)-(3.2) are not periodic and tend to the boundary of* $(0, u_1) \times (0, v_1)$.

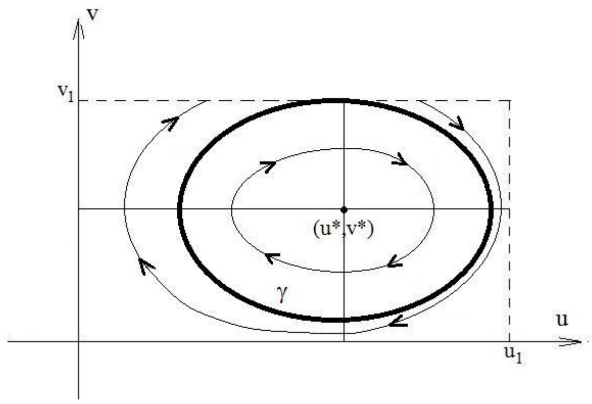

Figure 2: The curve γ

The following proposition describes curve γ more in details.

Proposition 3.1. *Consider the same hypotheses as in Theorem 3.2 and let*

$$c^{**} = \min\{\overline{A} + B(v^*), \overline{B} + A(u^*), \underline{A} + B(v^*), \underline{B} + A(u^*)\}.$$

By the assumptions, one has $c^* < c^{**} < +\infty$*. Consider the closure of the level set* $H(u, v) = c^{**}$ *in* $[0, u_1] \times [0, v_1]$*. Such a set is the trace of a closed curve* γ *which is contained in* $(0, u_1) \times (0, v_1)$ *except for at most four points (see Figure 2).*

4. Applying the model

System (3.1)-(3.2) represents a generalized version of the Goodwin model where the closed orbits are bounded within the $(0, u_1) \times (0, v_1)$ interval. It is usually assumed that $u_1 = v_1 = 1$, but different values are possible, provided that they are positive and lower than unity.

As stated above, assumptions 1., 2. and 3. are quite general and, under proper values of the parameters, they are satisfied even by the basic Goodwin model. By contrast, Assumption 4., which requires f, g, ϕ and ψ to have functional forms allowing the divergence of $A(u)$ and $B(v)$ on the boundaries, is not fulfilled in $u = u_1$ and $v = v_1$ by the original Goodwin model. In fact, in this case f and g are identically equal to one and both the investment function, $(1 - u)/\sigma$, and the real wage bargaining function, $-\gamma + \rho v$, are linear and defined for all $u, v > 0$, so that $A(u)$ and $B(v)$ diverge at $u = v = 0^+$, but they do not diverge elsewhere.

The generalized model presented in Section 3 includes the extensions of the Goodwin model proposed by Desai *et al* [6] and Harvie *et al* [14] (see Section 2) as special cases.

As for the model by Desai *et al* (see system (2.3)-(2.4)), we have:

$$v_1 = 1, \qquad u_1 = \bar{u},$$

$$f(u) = g(v) = 1,$$

$$\psi(v) = -(\gamma + \alpha) + \rho(1 - v)^{-\delta},$$

and

$$\phi(u) = (\lambda \log(1 - \bar{u}) + (\alpha + \beta)) - \lambda \log(\bar{u} - u).$$

Under the assumptions made by the authors (see Section 2), conditions 1. to 3. are readily fulfilled. Since

$$A(u) = \int \frac{(\lambda \log(1 - \bar{u}) + (\alpha + \beta)) - \lambda \log(\bar{u} - u)}{u}\, du,$$

and

$$B(v) = \int \frac{-(\gamma + \alpha) + \rho(1 - v)^{-\delta}}{v}\, dv,$$

$A(0^+)$ and $B(0^+)$ diverge as required by Assumption 4., while for $B(1^-)$ to diverge we have to impose the additional condition $\delta \geq 1$. As for $A(u)$, the presence of a logarithmic investment function prevents this from diverging in $\bar{u}$. A possible alternative would be a polynomial investment function with powers greater or equal than unity, like the functional form the authors have used for the real wage bargaining function.

Thus, the conditions imposed by Desai *et al* are not sufficient to guarantee that all trajectories lie within the $(0,\bar{u})\times(0,1)$ interval. More precisely, our discussion in Section 3 ensures that when $c > \min\{\overline{A} + B(v^*), \overline{B} + A(u^*)\}$ the system has no periodic solutions in the $(0,\bar{u})\times(0,1)$ interval: for these values of c every solution will reach the boundary.

As for the model by Harvie *et al* (see system (2.5)-(2.6)), one has:

$$u_1 = 1, \qquad v_1 = 1,$$

$$f(u) = k_1 u^{\mu_1}(1-u)^{\eta_1},$$

$$g(v) = k_2 v^{\mu_2}(1-v)^{\eta_2},$$

$$\psi(v) = -(\alpha+\gamma) + \rho v,$$

and

$$\phi(u) = -\frac{1}{\sigma} + (\alpha+\beta) + \frac{1}{\sigma}u.$$

Under the authors' assumptions on the size of the parameters involved in the original Goodwin model (see Section 2), conditions 1. to 3. are readily fulfilled. In this case

$$A(u) = \int \frac{\alpha+\beta-(1-u)/\sigma}{k_1 u^{\mu_1+1}(1-u)^{\eta_1}}\, du,$$

and

$$B(v) = \int \frac{-(\alpha+\gamma)+\rho v}{k_2 v^{\mu_2+1}(1-v)^{\eta_2}}\, dv.$$

For $A(u)$ to diverge to $+\infty$ as u goes to zero, we have to impose $k_1 > 0$ and $\mu_1 \geq 0$; while for it to diverge as u goes to one we also need $\eta_1 \geq 1$. Likewise, for $B(v)$ to fulfill Assumption 4. we need $k_2 > 0$, $\mu_2 \geq 0$ and $\eta_2 \geq 1$. The authors, however, impose $\eta_1 > 0$ and $\eta_2 > 0$, but these conditions are not sufficient to guarantee that all trajectories lie inside the unit box. More precisely, if one of the two exponents η_1 or η_2 lies between 0 and 1, then either $A(1^-)$ or $B(1^-)$ is finite and our discussion in Section 3 ensures that for sufficiently large values of c the system has no periodic solutions for every choice of the initial condition.

5. Analysis of the period

Now we turn to the analysis of the period of the cycles generated by system (3.1)-(3.2). First, we analyze the period length near the equilibrium point, obtaining a general result which is well-known in the special case of the standard Lotka-Volterra/Goodwin system and may clarify the use of the period of the linearized system in the empirical studies. Then, we analyze the period of the cycles close to the boundaries of the $(0, u_1) \times (0, v_1)$ interval. In this case, the result we obtain appears less obvious and it may help to explain some of the empirical evidence on the Goodwin model.

5.1. Period of small cycles

In order to simplify the analysis, we consider a system with a center at $(0, 0)$. We can always reduce ourselves to this case by a change of variables (see Corollary 5.1). We thus consider the bidimensional autonomous system

$$z' = F(z),$$

where $z = (u, v) \in \mathbb{R}^2$, $F(z) = F(u, v) = (F_1(u, v), F_2(u, v))$.

We obtain the following

Proposition 5.1. *Let $F : D \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a function of class C^1 in D, where D is an open set containing 0, and suppose that $F(0) = 0$. Assume that*

- $F_1(u, v)v > 0, \qquad F_2(u, v)u < 0, \qquad \forall (u, v) \in D \setminus \{0\};$
- *every solution which has initial conditions in D is a periodic solution;*
- *the linearized system*
$$z' = F'(0)z$$
has a periodic solution of minimal period $\tau > 0$.

Then, denoting by τ_s the period of the solution of system $z' = F(z)$ with starting point $(s, 0)$, $s > 0$, one has

$$\lim_{s \to 0} \tau_s = \tau.$$

Proof: Denote by $z_s(t)$ the solution of the Cauchy problem

$$\begin{cases} z' = F(z), \\ z(0) = (s, 0), \end{cases}$$

with $(s,0)\in D$, and consider $w_s(t)=\frac{z_s(t)}{\|z_s\|_\infty}$. One has $\rho_s=\|z_s\|_\infty\to 0$ as $s\to 0$. In fact, the solution z_s intersects the v-axis at points $(0,s^1)$ and $(0,s^3)$, $s^1<0<s^3$, and the u-axis at point $(s^2,0)$, $s^2<0<s$. One has $s^1\to 0$, otherwise there exists a sequence $\{s_k\}$ converging to 0 such that $s_k^1\to\hat{s}<0$. In this case, the solution of the problem with initial conditions $u(0)=0$, $v(0)=\frac{\hat{s}}{2}$ intersects the orbit of one of the solutions z_s, contradicting the unicity of the solutions. Similarly, $s^2\to 0$ and $s^3\to 0$. Since the conclusions of our discussion in Section 3 are still valid under our assumptions, the orbit of the solution is contained in the box $[s^2,s]\times[s^1,s^3]$, so that $\rho_s\to 0$ as $s\to 0$.

Since $\|w_s\|_\infty=1$, the sequence $w_s(t)$ is equibounded. In addition, we have

$$w_s'(t)=\frac{F(z_s(t))}{\rho_s}=\frac{F(w_s(t)\rho_s)}{\rho_s}. \tag{5.1}$$

Since the derivative of $F(z)$ is bounded in a neighborhood of 0, it is also Lipschitzian in such a neighborhood. Hence, there exists $L>0$ such that

$$\frac{F(w_s(t)\rho_s)}{\rho_s}=\frac{F(w_s(t)\rho_s)-F(0)}{\rho_s}\le L\frac{\|w_s(t)\rho_s\|_\infty}{\rho_s}=L\|w_s(t)\|_\infty=L.$$

Thus we have that for s small enough, the set of functions $\{w_s(\cdot)\}$ has a bounded derivative, hence it is equicontinuous. We can therefore apply the Ascoli-Arzelá Theorem and conclude that there exists a sequence w_{s_k} such that w_{s_k} converges uniformly on the compact sets of $[0,+\infty)$ to a continuous function $v(t):[0,+\infty)\to\mathbb{R}$. By (5.1) we get

$$w_{s_k}(t)=w_{s_k}(0)+\int_0^t\frac{F(w_{s_k}(\xi)\rho_{s_k})}{\rho_{s_k}}d\xi.$$

Since $\frac{F(w_{s_k}(t)\rho_{s_k})}{\rho_{s_k}}$ converges to $F'(0)v(t)$ uniformly on the compact sets of $[0,+\infty)$, we can conclude that

$$v(t)=v(0)+\int_0^t F'(0)v(\xi)d\xi,$$

and thus $v(t)$ is a (non-trivial) solution of the linear system

$$v'(t)=F'(0)v(t).$$

Let $\epsilon>0$. Since $v(t)$ has (minimal) period τ, by Kamke's theorem, for s small enough, $w_s(t)$ has two zeros in the time interval $(0,\tau+\epsilon)$, therefore $\tau_s<\tau+\epsilon$ (see also Remark 3.1). Passing to subsequences if necessary, we can assume that $\tau_{s_k} \to \tau^*$. By the periodicity of $w_{s_k}(t)$ one has $w_{s_k}(t + \tau_{s_k}) = w_{s_k}(t)$, hence $v(t + \tau^*) = v(t)$ and therefore $\tau^* \geq \tau$. Hence, $\tau \leq \tau^* \leq \tau + \epsilon$ and we can conclude that $\tau = \tau^*$.

By the above result, for every sequence $s_k \to 0$ there exists a subsequence for which the period converges to the period of the linearized system. Therefore $\tau_s \to \tau$ for $s \to 0$. □

If we now turn to system (3.1)-(3.2), using the above result and considering the change of variables $\tilde{u} = u - u^*$, $\tilde{v} = v - v^*$, we obtain the following

Corollary 5.1. *Suppose that assumptions 1., 2. and 3. in Section 3 are fulfilled and that $f, g \in C^1$. Denote by τ_s the period of the solution of system (3.1)-(3.2) with starting point $(u^*, v^* + s)$. Then $\lim_{s\to 0} \tau_s = \tau$, where τ is the period of the non-trivial solutions of the linear system*

$$\begin{aligned} \dot{u} &= \psi'(v^*)v, \\ \dot{v} &= -\phi'(u^*)u. \end{aligned}$$

This result is quite standard in the field of dynamical systems. In particular, Volterra himself ([22], [23]) had already proved this for the basic Lotka-Volterra/Goodwin system. Proposition 5.1, however, is more general and it may be of some theoretical interest even beyond the specific applications considered here. Moreover, it has some interesting implications from an empirical point of view. In fact, in almost every contribution to the empirical literature on the Goodwin model (see for instance Harvie [13]) the period length of the cycles is approximated by the period of the linearized system, which is usually easier to compute. The above result guarantees that this approximation is valid near the fixed point and that this holds even for much more general systems.

5.2. Period of large cycles

Consider again the system

$$\begin{aligned} \dot{u} &= uf(u)\psi(v), \\ \dot{v} &= -vg(v)\phi(u). \end{aligned}$$

In Section 3 we proved that, under conditions 1., 2., 3. and 4., system (3.1)-(3.2) has a periodic solution for every $c > c^*$, where $c^* = H(u^*, v^*)$. We denote by τ_c such a period, which is the same for every choice of the starting point lying on the level line $H(u, v) = c$.

Now consider the additional conditions

5a.

$$f, g \in C^1, \quad \lim_{v \to 0^+} v g(v) = 0^+;$$

5b.

$$f, g \in C^1, \quad \lim_{u \to 0^+} u f(u) = 0^+.$$

We then have the following

Theorem 5.1. *Assume that conditions 1., 2., 3. and 4. in Section 3 are satisfied and that either 5a. or 5b. holds. Then*

$$\lim_{c \to +\infty} \tau_c = +\infty.$$

Proof: We assume that Assumption 5a. is satisfied (the proof is analogous when 5b. holds). Consider the Cauchy problem

$$\begin{cases} \dot{u} = u f(u) L_2, \\ u(0) = u_0, \end{cases} \tag{5.2}$$

where $u_0 \in (0, u_1)$. The solution $u(t)$ is unique (since $f \in C^1(0, u_1)$), decreasing and positive. One has

$$\frac{\dot{u}(t)}{u(t) f(u(t))} = L_2,$$

and, integrating,

$$\int_0^t \frac{\dot{u}(s)}{u(s) f(u(s))} ds = L_2 t.$$

By the change of variable $\xi = u(t)$ we get

$$\int_{u_0}^{u(t)} \frac{d\xi}{\xi f(\xi)} = L_2 t.$$

Note that $\int_0^x \frac{d\xi}{\xi f(\xi)} = +\infty$ for all $x \in (0, u_1)$. In fact, by Assumption 4. $\int_0^x \frac{\phi(\xi)}{\xi f(\xi)} d\xi = -\infty$ and by Assumption 3. $\lim_{\xi \to 0} \phi(\xi) = L_1 < 0$.

Let $[0, \hat{t})$ be the maximal interval of existence of $u(t)$. If $\hat{t} < +\infty$ then $\int_{u(\hat{t})}^{u_0} \frac{d\xi}{\xi f(\xi)} < +\infty$, therefore $u(\hat{t}) > 0$. Since $\hat{t} \in \mathbb{R}$ and $u(\hat{t}) > 0$, by the existence and unicity of the solution we have that the solution of (5.2) is prolongable after $\hat{t}$, in contradiction to the definition of $\hat{t}$.

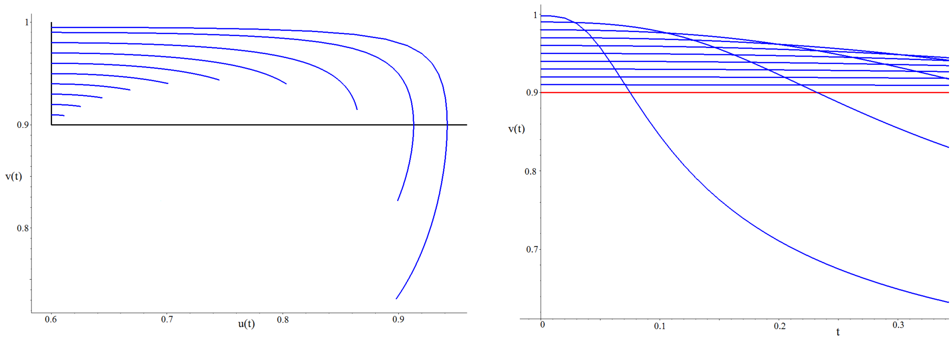

Figure 3: First quadrant: solution trajectories in the (u, v) phase plane for different initial values of v (left) and the corresponding time paths of v for $t \in [0, 0.35]$ (right).

We have proved that $\hat{t} = +\infty$ and $u(t) \to 0$ as $t \to +\infty$.

Now let $T > 0$ and consider the solution $u(t)$ of the Cauchy problem (5.2) defined over the time interval $[0, T]$. Since $\lim_{v \to 0^+} vg(v) = 0^+$, we can define $vg(v)$ at $v = 0$. In this way $(u(t), 0)$ becomes a solution of the system

$$\begin{aligned} \dot{u} &= uf(u)L_2, \\ \dot{v} &= 0, \end{aligned}$$

satisfying the initial conditions $u(0) = u_0$, $v(0) = 0$.

Since system (3.1)-(3.2) satisfies the assumptions of Kamke's theorem (with $(u, v) \in (0, u_1) \times [0, v_1)$), for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $v_0 \in (0, \delta)$ the solution of system (3.1)-(3.2), with initial condition $u(0) = u_0$,

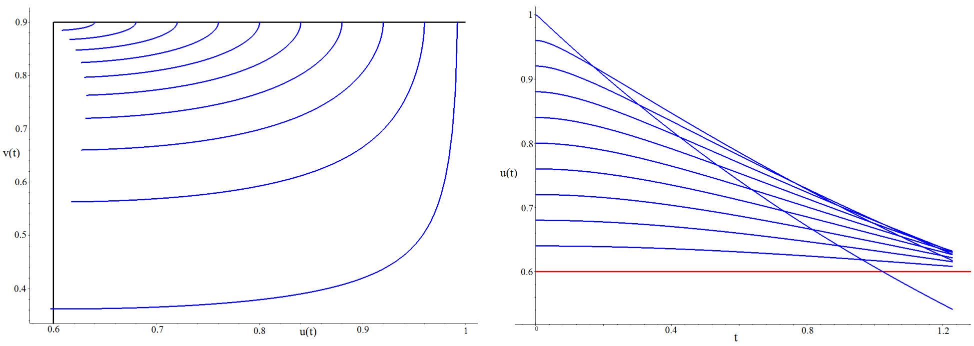

Figure 4: Fourth quadrant: solution trajectories in the (u, v) phase plane for different initial values of u (left) and the corresponding time paths of u for $t \in [0, 1.23]$.

$v(0) = v_0$ is defined over $[0, T]$ and, in addition,

$$|v(t)| < \epsilon, \quad \forall t \in [0, T].$$

Taking $\delta, \epsilon < v^*$, we can conclude that the period of the solution with starting point (u_0, v_0) is larger than T.

Let $c(\delta) = H(u_0, \frac{\delta}{2})$. If $c > c(\delta)$ then there exists v_0^c such that $H(u_0, v_0^c) = c$ and $v_0^c < \frac{\delta}{2}$. Therefore $\tau_c > T$. This proves that $\lim_{c\to+\infty} \tau_c = +\infty$. □

The above result has some interesting implications. It shows that as we approach the frontier of the feasible interval and consider very large cycles, the period length tends to infinity. More precisely, we have proved that this tendency is due to the fact that the trajectories are passing near the origin (which is always a saddle point).

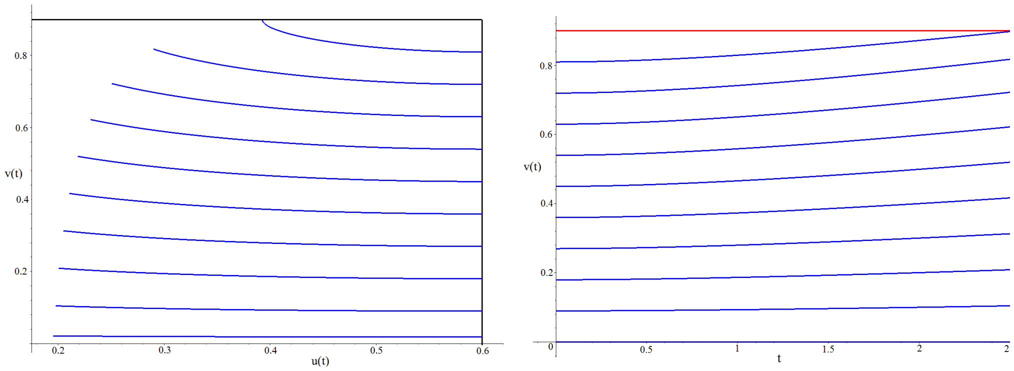

Figure 5: Third quadrant: solution trajectories in the (u, v) phase plane for different initial values of v (left) and the corresponding time paths of v for $t \in [0, 2.5]$ (right).

Figures 3 to 7 show the results of a simulation based upon the model proposed by Desai *et al* [6] (we adopted the same parameter values) where, however, we replaced their logarithmic investment function (which, as shown in Section 4, does not guarantee that all the solution trajectories lie within the feasible interval) with one of the form $-\frac{(1-u)^{-\delta}}{\sigma}$ with δ =1.2. We considered the fixed point (u^*, v^*) =(0.6, 0.9). Figures 3 to 6 show the solution trajectories in each single quadrant after a specified time interval, while Figure 7 shows the whole period length as a function of the starting point. From this last figure we see that the period length is very similar to that of the linearized system when we consider starting points close to the center. The period increases as we consider starting points far from the center and it eventually explodes as we approach the frontier of the unit box. Figure

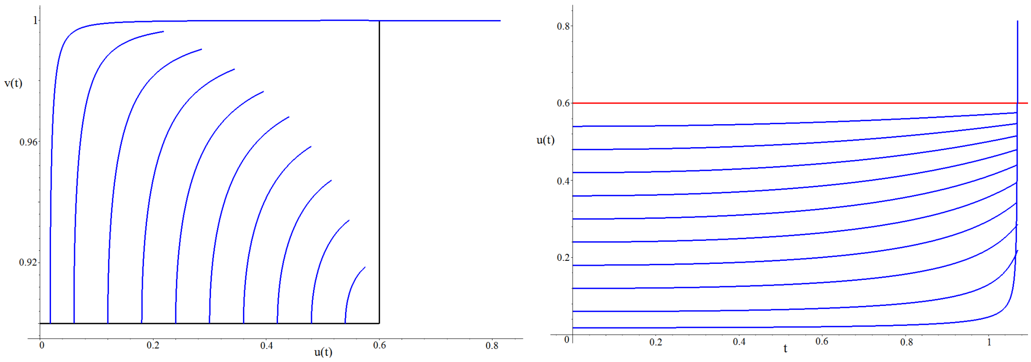

Figure 6: Second quadrant: solution trajectories in the (u, v) phase plane for different initial values of u (left) and the corresponding time paths of u for $t \in [0, 1.06]$ (right).

7 supports the idea that the period is increasing monotonic (it increases as we take starting points closer to the frontier), but such a result has not been proved rigorously and it requires further research. In fact, while the monotonicity of the period has been proved in the special case of the basic Lotka-Volterra/Goodwin model (see for instance Waldvogel [24]), it appears that no result in the literature can be applied to a general model like system (3.1)-(3.2).

Our numerical simulations show that in the case of our modified version of the model by Desai *et al* [6], the fact that the period is increasing is mainly due to its length in the third quadrant (low u and low v), which is larger than that in the other three quadrants and is increasing. The opposite holds in the first quadrant (high u and high v), where the speed of the solution trajectories increases as we approach the frontier. The results are mixed in the second and fourth quadrant. Note that while the whole period is monotonic, this is not true at all in the case of the single quadrants. This shows the difficulty of proving a general theorem on the monotonicity of the period.

The above results may help to explain some of the empirical evidence on the Goodwin model, which suggests the presence of a three-quarter cycle in many OECD countries, starting from low values of u and high values of v and ending with high values of u and low values of v (see Harvie [13]). In fact, the missing quarter of the cycle corresponds to the region where, as in the model considered in our simulation, the solution trajectories may become very slow. Thus, it is plausible that it will take some years before the observed cycles close. Note that this is an unfavorable result from a socio-economic point of view: it means that the employment proportion will rise very slowly after a period of high unemployment.

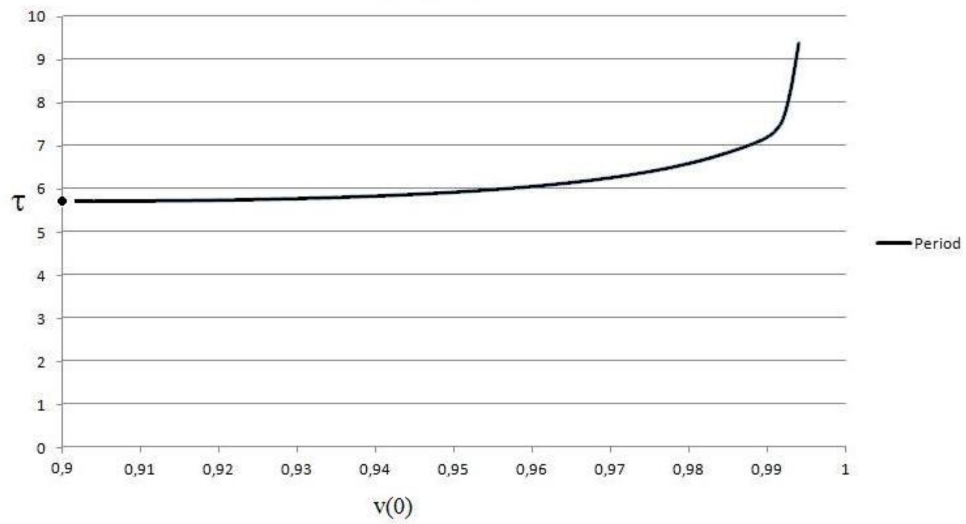

Figure 7: Period length as a function of the initial point $(0, v(0))$, with $v(0)$ from 0.904 to 0.994. τ is the period of the linearized system and it equals 5.697.

6. Introducing inflation

Many authors over the years have extended the basic Goodwin model by introducing economic phenomena which had not been taken into account by Goodwin. One of these phenomena is inflation. This has often been modelled introducing an additive term in the real wage bargaining function. This term is an increasing function of the wage share u, since firms are assumed to set their prices as a mark up over the unit labor cost of output (see for instance Desai [4]; van der Ploeg [20]; Flaschel [9]). The additive term is then multiplied by a constant, denoted by η, which in the case $\eta > 0$ measures the degree of money illusion. In this section we introduce inflation in system (3.1)-(3.2) by considering a model similar to those proposed by the literature, but in the context of our generalized approach. First, we study how the introduction of inflation affects the stability of the equilibrium point. Then, we provide a condition to determine whether the equilibrium point is a focus or a node.

6.1. Stability of the equilibrium

Consider the system

$$\frac{\dot{u}}{u} = f(u)(\psi(v) - \eta h(u)), \tag{6.1}$$

$$\frac{\dot{v}}{v} = -g(v)\phi(u), \tag{6.2}$$

where f, g, ψ, ϕ satisfy assumptions 1., 2., 3. and 4. of Section 3, η is a real parameter, $h(u) : (0, u_1) \to \mathbb{R}$ is C^1 and $h' > 0$. As already remarked, there exists a unique value u^* such that $\phi(u^*) = 0$. Assume that $\eta h(u^*) \in Im(\psi)$ and let v^* be the unique solution of $\psi(v^*) = \eta h(u^*)$, so that (u^*, v^*) is the unique fixed point of system (6.1)-(6.2).

Setting $\tilde{\psi}(v) = \psi(v) - \psi(v^*)$ and $\tilde{h}(u) = h(u) - h(u^*)$, one has that equation (6.1) becomes

$$\begin{aligned} \frac{\dot{u}}{u} &= f(u)(\psi(v) - \eta h(u)) = f(u)(\psi(v) - \psi(v^*) + \psi(v^*) - \eta h(u)) \\ &= f(u)(\psi(v) - \psi(v^*) + \eta h(u^*) - \eta h(u)) = f(u)(\tilde{\psi}(v) - \eta \tilde{h}(u)), \end{aligned}$$

and thus

$$\frac{\dot{u}}{u} = f(u)\tilde{\psi}(v) - \eta f(u)\tilde{h}(u).$$

Note that

$$\tilde{\psi}(v^*) = 0, \quad \tilde{h}(u^*) = 0.$$

We set

$$A(u) = \int_{u^*}^{u} \frac{\phi(\xi)}{\xi f(\xi)} d\xi, \qquad \tilde{B}(v) = \int_{v^*}^{v} \frac{\tilde{\psi}(\xi)}{\xi g(\xi)} d\xi.$$

We have

$$\frac{\dot{u}\phi(u)}{u f(u)} = \phi(u)\tilde{\psi}(v) - \eta\phi(u)\tilde{h}(u),$$

$$\frac{\dot{v}\tilde{\psi}(v)}{v g(v)} = -\phi(u)\tilde{\psi}(v).$$

Therefore

$$\frac{d}{dt}[A(u(t) + \tilde{B}(v(t))] = -\eta\phi(u)\tilde{h}(u).$$

In the sequel we use LaSalle's principle in order to prove the stability of the fixed point $z^* = (u^*, v^*)$ when $\eta > 0$. For this purpose we need the concept of Liapunov function which, although classical, is recalled here for the reader's convenience in the context of LaSalle's principle.

Let $\Omega \subset \mathbb{R}^n$ be an open set and let $F : \Omega \to \mathbb{R}^n$ be a continuous vector field. We assume the uniqueness of the solutions of the Cauchy problems associated to

$$z' = F(z) \tag{6.3}$$

and denote by $z(t, x)$ the solution of (6.3) with $z(0) = x \in \Omega$. We also assume that there is a unique equilibrium point $z^* \in \Omega$ (so that $F(z^*) = 0$). $V : \Omega \to \mathbb{R}$, is a Liapunov function if it is positive definite, i.e. $V(x) > V(z^*) = 0$, $\forall x \neq z^*$, and

$$\dot{V}(x) = \frac{d}{dt} V(z(t, x))|_{t=0} \leq 0, \quad \forall x \in \Omega.$$

Here we assume that $V(x)$ is a C^1 function, so that $\dot{V}(x) = \langle \nabla V(x), F(x) \rangle$ for all $x \in \Omega$.

LaSalle's principle can be stated as follows (see LaSalle [15]).

Theorem 6.1 (LaSalle's principle). *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $V : \Omega \to \mathbb{R}$ be a positive definite Liapunov function on Ω. Suppose that for some $c > 0$ the set $\Omega_c = \{x \in \Omega : \ V(x) \leq c\}$ is a non-empty compact set. We define*

$$S = \{x \in \Omega_c : \ \dot{V}(x) = 0\}.$$

Then, for every initial point $z_0 \in \Omega_c$ the solution $z(t, z_0)$ tends, as $t \to +\infty$, to the largest invariant set inside S. In particular, if S contains no invariant sets other than $x = z^*$, then z^* is asymptotically stable.

In our case we define $V(u, v) = A(u) + \tilde{B}(v)$. Since $\phi(u)$ and $\tilde{h}(u)$ are strictly increasing and they both vanish at u^*, we conclude that for $\eta > 0$

$$\frac{d}{dt}[V(u(t), v(t)] \leq 0$$

for every trajectory, so that $V(u, v)$ is a Liapunov function. The set S is given by $\{(u, v) \in \mathbb{R}^2 : \ u = u^*\} \cap \Omega$ and the only invariant set in S is $\{u^*, v^*)$. Therefore, recalling that (as proved in Section 3) for every $c > 0 = V(u^*, v^*)$ the set $\{(u, v) : V(u, v) \leq c\}$ is compact, we can apply LaSalle's principle. We thus conclude that

1. if $\eta > 0$, (u^*, v^*) is a globally asymptotically stable equilibrium point;

2. if $\eta < 0$, (u^*, v^*) is a globally repulsive equilibrium point.

The above result shows that system (3.1)-(3.2), like the basic Goodwin model, is structurally unstable, i.e. small perturbations of the model may affect the qualitative behavior of the solutions. In the literature on the Goodwin model the presence of inflation, combined with a certain degree of money illusion ($\eta > 0$), has a stabilizing effect on the economy. Our result shows that this effect occurs even in the context of a much more general framework and, in addition, it holds globally.

6.2. Focus or node

Now we turn to the issue of determining whether the equilibrium point (u^*, v^*) is a focus or a node. We consider the same assumptions as above.

Consider the polar coordinates

$$\begin{cases} u(t) - u^* = \rho(t)\cos\theta(t), \\ v(t) - v^* = \rho(t)\sin\theta(t), \end{cases}$$

so that

$$\dot{u} = \dot{\rho}\cos\theta - \rho\dot{\theta}\sin\theta, \tag{6.4}$$

$$\dot{v} = \dot{\rho}\sin\theta + \rho\dot{\theta}\cos\theta. \tag{6.5}$$

Let us multiply (6.4) by $\rho\sin\theta$ and (6.5) by $-\rho\cos\theta$. Summing the two terms we get

$$\dot{u}\rho(\sin\theta) - \dot{v}\rho(\cos\theta) = -\rho^2\dot{\theta},$$

hence

$$-\dot{\theta} = \frac{\dot{u}(v - v^*) - \dot{v}(u - u^*)}{\rho^2},$$

from which

$$-\dot{\theta} = \frac{uf(u)(\psi(v) - \eta h(u))(v - v^*) + vg(v)\phi(u)(u - u^*)}{(u - u^*)^2 + (v - v^*)^2}.$$

Evaluating the differential of the second term at point (u^*, v^*) we get

$$-\dot{\theta} = \frac{\mathcal{N}(u - u^*, v - v^*)}{(u - u^*)^2 + (v - v^*)^2} + \frac{o(u - u^*, v - v^*)}{(u - u^*)^2 + (v - v^*)^2},$$

where

$$\mathcal{N}(u - u^*, v - v^*) =$$
$$v^*g(v^*)\phi'(u^*)(u - u^*)^2 - \eta u^* f(u^*)h'(u^*)(u - u^*)(v - v^*) + u^* f(u^*)\psi'(v^*)(v - v^*)^2.$$

The numerator $\mathcal{N}$ is a quadratic form whose associated matrix is

$$\begin{pmatrix} v^* g(v^*)\phi'(u^*) & -\frac{\eta}{2}u^* f(u^*)h'(u^*) \\ -\frac{\eta}{2}u^* f(u^*)h'(u^*) & u^* f(u^*)\psi'(v^*) \end{pmatrix}. \tag{6.6}$$

If such a form is positive definite then there exists $k > 0$ such that for ρ small enough one has

$$-\dot{\theta} \geq \frac{k\rho^2}{\rho^2} = k. \tag{6.7}$$

Since the diagonal elements in (6.6) are positive, it is sufficient to consider the determinant, which is given by

$$u^* v^* f(u^*)g(v^*)\phi'(u^*)\psi'(v^*) - \frac{\eta^2}{4}(u^*)^2 f(u^*)^2 h'(u^*)^2.$$

Hence the quadratic form is positive definite when

$$v^* g(v^*)\phi'(u^*)\psi'(v^*) > \frac{\eta^2}{4}u^* f(u^*)h'(u^*)^2,$$

i.e. when

$$\eta^2 < \frac{4v^* g(v^*)\phi'(u^*)\psi'(v^*)}{u^* f(u^*)h'(u^*)^2}. \tag{6.8}$$

In view of (6.7), we conclude that when condition (6.8) is satisfied the equilibrium point (u^*, v^*) is a focus. By contrast, if the reverse strict inequality holds in (6.8), the quadratic form is indefinite and the equilibrium point is a node.

The above result suggests that when the wage setting process is characterized by a low degree of money illusion (i.e. η is positive and small), the economy will fluctuate while approaching the equilibrium point. By contrast, if money illusion increases beyond a certain level the economy will finally converge monotonically to the equilibrium.

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