Noncooperative Oligopoly in Markets with a Cobb-Douglas Continuum of Traders

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Abstract

In this paper, we reconsider two models of noncooperative oligopoly in general equilibrium based on a particular strategic market game, the so called Shapley's window model, introduced by Busetto et al. (2008), (2011) under the assumption that preferences of the traders belonging to the atomless part are represented by Cobb-Douglas utility functions. First, we prove the existence of a Cournot-Nash equilibrium. Then, we show that the set of the Cournot-Walras equilibrium allocations is a subset of the set of the Cournot-Nash equilibrium allocations. Finally, we partially replicate the exchange economy by increasing the number of atoms without affecting the atomless part while ensuring that the measure space of agents remains finite. We show that any sequence of Cournot-Nash equilibrium allocations of the strategic market game associated with the partially replicated exchange economies approximates a Walras equilibrium allocation of the original exchange economy.

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1 Introduction

Noncooperative oligopoly in interrelated markets has been modeled in two main directions. The first is the strategic market games approach, developed by Shapley and Shubik (see also Dubey and Shubik (1977), Postlewaite and Schmeidler (1978), Okuno et al. (1980), Mas-Colell (1982), Sahi and Yao (1989), Amir et al. (1990), Peck et al. (1992), Dubey and Shapley (1994), among others). In this class of models, all traders behave strategically and prices are determined according to non-Walrasian pricing rules. The second is the Cournot-Walras approach, developed by Gabszewicz and Vial (1972) for economies with production (see also Roberts and Sonnenschein (1977), Roberts (1980), Mas-Colell (1982), Dierker and Grodal (1986), among others), and by Codognato and Gabszewicz (1991) for pure exchange economies (see also Codognato and Gabszewicz (1993), d'Aspremont et al. (1997), Gabszewicz and Michel (1997), Shitovitz (1997), Julien and Tricou (2005), (2009), among others). In this class of models, some agents behave strategically while others behave competitively and prices are determined according to the Walrasian pricing rule. Then, strategic agents determine their strategies as in the Cournot game (see Cournot (1838)) taking into account the Walrasian price correspondence. Both classes of models aim at studying the working and the consequences of market power in a general equilibrium framework.

More recently, Busetto et al. (2008), (2011) introduced two models of noncooperative oligopoly in general equilibrium based on the Shapley's window model. This model was originally proposed by Lloyd S. Shapley and further analyzed by Sahi and Yao (1989) in exchange economies with a finite number of traders, and Codognato and Ghosal (2000) in exchange economies with an atomless continuum of traders. In particular, Busetto et al. (2011), taking inspiration from a seminal paper by Okuno et al. (1980), considered the Cournot-Nash equilibrium of the Shapley's window model associated with an exchange economy à la Shitovitz (see Shitovitz (1973)) with atoms and an atomless part, whereas Busetto et al. (2008) provided a respecification à la Cournot-Walras of this model assuming that atoms behave à la Cournot while the atomless part behaves à la Walras.

In this paper, we reconsider these two models under the assumption that preferences of the traders belonging to the atomless part are represented by Cobb-Douglas utility functions. Beyond their tractability to compute solutions in theoretical models, Cobb-Douglas utility functions are very useful to understand the relationships among equilibrium concepts.

We first show the existence of a Cournot-Nash equilibrium under the following set of assumptions: (i) each trader is endowed with a strictly positive amount of at least one commodity and each commodity is held, in the aggregate, by the atomless part; (ii) atoms' utility functions are continuous, strongly monotone, and quasi-concave; (iii) traders' utility functions are jointly measurable. Busetto et al. (2011) proved the existence of a Cournot-Nash equilibrium under less restrictive assumptions on the atomless part's endowments and preferences. In particular, they assumed that the atomless part has continuous, strongly monotone, and quasi-concave preferences without requiring that it holds, in the aggregate, each commodity. Nevertheless, our proof is not a special case of theirs as they had to impose, following Sahi and Yao (1989), a further restrictive assumption on atoms, namely that there exists at least two atoms with endowments and indifference curves contained in the strict interior of the commodity space. Therefore, our proof allows dealing with cases where all atoms have corner endowments and indifference curves which cross the boundary of the commodity space.

Then, we provide, following Busetto et al. (2008), a respecification à la Cournot-Walras of our model and we prove prove that, under the same assumptions of our existence theorem, the set of the Cournot-Walras equilibrium allocations is a subset of the Cournot-Nash equilibrium allocations. Busetto et al. (2008) provided an example which shows that this result may not hold if preferences of the traders belonging to the atomless part are not represented by Cobb-Douglas utility functions.

Finally, we consider the limit relationship between the Cournot-Nash equilibrium allocations and the Walras equilibrium allocations of our model. Busetto et al. (2012) proved a limit result under the same assumptions of their existence theorem. Here, we use the same kind of replication they proposed, namely, we partially replicate the exchange economy by increasing the number of atoms without affecting the atomless part while ensuring that the measure space of agents is finite. We show that, under the same assumptions which sustain our existence theorem, any sequence of Cournot-Nash equilibrium allocation of the Shapley's window model associated with the partially replicated exchange economy approximates the Walras equilibrium allocation of the original exchange economy. Our proof and that provided by Busetto et al. (2008) differ as they are drawn from different sufficient conditions.

The paper is organized as follows. In Section 2, we present the mathematical model and state the main assumptions. In Section 3, we show

the existence of the Cournot-Nash equilibrium. Section 4 is devoted to the Cournot-Walras equilibrium. Section 5 aims at studying the relationship between the Cournot-Nash and the Cournot-Walras equilibrium. In Section 6, we show the limit relationship between the Cournot-Nash and the Walras equilibrium.

2   The mathematical model

We consider a pure exchange economy, $\mathcal{E}$, with large traders, represented as atoms, and small traders, represented by an atomless part. The space of traders is denoted by the measure space $(T, \mathcal{T}, \mu)$, where T is the set of traders, $\mathcal{T}$ is the σ-algebra of all μ-measurable subsets of T, and μ is a real valued, non-negative, countably additive measure defined on $\mathcal{T}$. We assume that $(T, \mathcal{T}, \mu)$ is finite, i.e., $\mu(T) < \infty$. This implies that the measure space $(T, \mathcal{T}, \mu)$ contains at most countably many atoms. Let T_1 denote the set of atoms and $T_0 = T \setminus T_1$ the atomless part of T. A null set of traders is a set of measure 0. Null sets of traders are systematically ignored throughout the paper. Thus, a statement asserted for "each" trader in a certain set is to be understood to hold for all such traders except possibly for a null set of traders. The word "integrable" is to be understood in the sense of Lebesgue.

There are l different commodities. A commodity bundle is a point in R_+^l. An assignment (of commodity bundles to traders) is an integrable function $\mathbf{x}$: $T \to R_+^l$. There is a fixed initial assignment $\mathbf{w}$, satisfying the following assumption.

Assumption 1. $\mathbf{w}(t) > 0$, for each $t \in T$, $\int_{T_0} \mathbf{w}(t) \, d\mu \gg 0$.

Furthermore, as in Sahi and Yao (1989), we can assume, for convenience, that $\int_T \mathbf{w}^j(t) \, d\mu = 1$, $j = 1, \ldots, l$. An allocation is an assignment $\mathbf{x}$ for which $\int_T \mathbf{x}(t) \, d\mu = \int_T \mathbf{w}(t) \, d\mu$. The preferences of each trader $t \in T$ are described by a utility function $u_t : R_+^l \to R$, satisfying the following assumptions.

Assumption 2. $u_t : R_+^l \to R$ is continuous, strongly monotone, and quasi-concave, for each $t \in T_1$, $u_t(x) = x^{1\boldsymbol{\alpha}^1(t)} \cdots x^{l\boldsymbol{\alpha}^l(t)}$, for each $t \in T_0$ and for each $x \in R_+^l$, where $\boldsymbol{\alpha}$: $T_0 \to R_{++}^l$ is a function such that $\boldsymbol{\alpha}^j(t) > 0$, $j = 1, \ldots, l$, $\sum_{j=1}^l \boldsymbol{\alpha}^j(t) = 1$, for each $t \in T_0$.

Let $\mathcal{B}(R_+^l)$ denote the Borel σ-algebra of R_+^l. Moreover, let $\mathcal{T} \otimes \mathcal{B}$ denote the σ-algebra generated by the sets $E \times F$ such that $E \in \mathcal{T}$ and $F \in \mathcal{B}$.

Assumption 3. $u : T \times R_+^l \to R$ given by $u(t,x) = u_t(x)$, for each $t \in T$ and for each $x \in R_+^l$, is $\mathcal{T} \bigotimes \mathcal{B}$-measurable.

A price vector is a vector $p \in R_+^l$. We define, for each $p \in R_+^l$, a correspondence $\boldsymbol{\Delta}_p : T \to \mathcal{P}(R^l)$ such that, for each $t \in T$, $\boldsymbol{\Delta}_p(t) = \{x \in R_+^l : px = p\mathbf{w}(t)\}$, a correspondence $\boldsymbol{\Psi}_p : T \to \mathcal{P}(R^l)$ such that, for each $t \in T$, $\boldsymbol{\Psi}_p(t) = \{x \in R_+^l : \text{ for all } y \in \boldsymbol{\Delta}_p(t), u_t(x) \geq u_t(y)\}$, and finally a correspondence $\mathbf{X}_p : T \to \mathcal{P}(R^l)$ such that, for each $t \in T$, $\mathbf{X}_p(t) = \boldsymbol{\Delta}_p(t) \cap \boldsymbol{\Psi}_p(t)$.

A Walras equilibrium of $\mathcal{E}$ is a pair $(p^*, \mathbf{x}^*)$, consisting of a price vector p^* and an allocation $\mathbf{x}^*$, such that $\mathbf{x}^*(t) \in \mathbf{X}_{p^*}(t)$, for each $t \in T$.

By Assumption 2, for each $p \in R_{++}^l$, it is possible to define the atomless part's Walrasian demands as a function $\mathbf{x}^0(\cdot, p) : T_0 \to R_+^l$ such that $\mathbf{x}^0(t, p) = \mathbf{X}_p$, for each $t \in T_0$. It is immediate to verify that $\mathbf{x}^{0j}(t, p) = \frac{\boldsymbol{\alpha}^j(t) \sum_{i=1}^l p^i \mathbf{w}^i(t)}{p^j}$, $j = 1, \ldots, l$, for each $t \in T_0$. The following proposition shows that this function is integrable.

Proposition 1. *Under Assumptions 1, 2, and 3, the function $\mathbf{x}^0(\cdot, p)$ is integrable, for each $p \in R_{++}^l$.*

Proof. Let $p \in R_{++}^l$. The restriction of $\mathbf{w}$ to T_0 is integrable as $\mathbf{w}$ is integrable. Now, we prove that $\boldsymbol{\alpha}$ is a measurable function. Consider a commodity bundle $y \in R_{++}^l$. Let $u^0(\cdot, y)$ denote the restriction of $u(\cdot, y)$ to T_0. The function $u(\cdot, y)$ must be measurable as, by Assumption 3, $u(\cdot, \cdot)$ is $\mathcal{T} \bigotimes \mathcal{B}$-measurable (see Theorem 4.48 in Aliprantis and Border (2006), p. 152). Then, the function $u^0(\cdot, y)$ is also measurable. Suppose that $\boldsymbol{\alpha}$ is not measurable. Then, there is an open set $O \in R_+^l$ such that $\boldsymbol{\alpha}^{-1}(O)$ is not a μ-measurable set. Let $f : R^l \to R_{++}^l$ be a function such that $f(v) = (y_1^{v_1}, \ldots, y_l^{v_l})$, for each $v \in R^l$. $f(O)$ is an open set as f is a homeomorphism. Suppose that $\tau \in \boldsymbol{\alpha}^{-1}(O)$. Then, $f(\boldsymbol{\alpha}(\tau)) \in f(O)$. But then, $\tau \in (u^0(\cdot, y))^{-1}(f(O))$. Therefore, $\boldsymbol{\alpha}^{-1}(O) \subset (u^0(\cdot, y))^{-1}(f(O))$. Suppose that $\tau \in (u^0(\cdot, y))^{-1}(f(O))$. Moreover, suppose that $\tau \notin \boldsymbol{\alpha}^{-1}(O)$. Then, $\boldsymbol{\alpha}(\tau) \notin O$. But then, $u^0(\tau, y) \notin f(O)$, a contradiction. Therefore, $(u^0(\cdot, y))^{-1}(f(O)) \subset \boldsymbol{\alpha}^{-1}(O)$. Then, $\boldsymbol{\alpha}^{-1}(O) = (u^0(\cdot, y))^{-1}(f(O))$. But then, $\boldsymbol{\alpha}^{-1}(O)$ is μ-measurable as $u^0(\cdot, y)$ is measurable, a contradiction. Therefore, $\boldsymbol{\alpha}$ is measurable. Hence, $\mathbf{x}^0(\cdot, p)$ is integrable as it is measurable and $\mathbf{x}^{0j}(t, p) < \frac{\sum_{i=1}^l p^i \mathbf{w}^i(t)}{p^j}$, $j = 1, \ldots, l$, for each $t \in T_0$. ∎

3   Cournot-Nash equilibrium

We introduce now the strategic market game, Γ, associated with $\mathcal{E}$. Let $b \in R_+^{l^2}$ be a vector such that $b = (b_{11}, b_{12}, \ldots, b_{ll-1}, b_{ll})$. A strategy correspondence is a correspondence $\mathbf{B} : T \to \mathcal{P}(R_+^{l^2})$ such that, for each $t \in T$, $\mathbf{B}(t) = \{b \in R_+^{l^2} : \sum_{j=1}^l b_{ij} \le \mathbf{w}^i(t), \, i = 1, \ldots, l\}$. A strategy selection is an integrable function $\mathbf{b} : T \to R^{l^2}$, such that, for each $t \in T$, $\mathbf{b}(t) \in \mathbf{B}(t)$. For each $t \in T$, $\mathbf{b}_{ij}(t)$, $i, j = 1, \ldots, l$, represents the amount of commodity i that trader t offers in exchange for commodity j. Given a strategy selection $\mathbf{b}$, we define the aggregate matrix $\bar{\mathbf{B}} = (\int_T \mathbf{b}_{ij}(t) \, d\mu)$. Moreover, we denote by $\mathbf{b} \setminus b(t)$ a strategy selection obtained by replacing $\mathbf{b}(t)$ in $\mathbf{b}$ with $b \in \mathbf{B}(t)$. With a slight abuse of notation, $\mathbf{b} \setminus b(t)$ will also represent the value of the strategy selection $\mathbf{b} \setminus b(t)$ at t.

Then, we introduce two further definitions (see Sahi and Yao (1989)).

Definition 1. *A nonnegative square matrix A is said to be irreducible if, for every pair (i, j), with $i \ne j$, there is a positive integer $k = k(i, j)$ such that $a_{ij}^{(k)} > 0$, where $a_{ij}^{(k)}$ denotes the ij-th entry of the k-th power A^k of A.*

Definition 2. *Given a strategy selection $\mathbf{b}$, a price vector p is market clearing if*

$$p \in R_{++}^l, \; \sum_{i=1}^l p^i \bar{\mathbf{b}}_{ij} = p^j \big(\sum_{i=1}^l \bar{\mathbf{b}}_{ji} \big), \, j = 1, \ldots, l. \tag{1}$$

By Lemma 1 in Sahi and Yao (1989), there is a unique, up to a scalar multiple, price vector p satisfying (1) if and only if $\bar{\mathbf{B}}$ is irreducible. Then, we denote by $p(\mathbf{b})$ a function which associates with each strategy selection $\mathbf{b}$ the unique, up to a scalar multiple, price vector p satisfying (1), if $\bar{\mathbf{B}}$ is irreducible, and is equal to 0, otherwise.

Given a strategy selection $\mathbf{b}$ and a price vector p, consider the assignment determined as follows:

$$\mathbf{x}^j(t, \mathbf{b}(t), p) = \mathbf{w}^j(t) - \sum_{i=1}^l \mathbf{b}_{ji}(t) + \sum_{i=1}^l \mathbf{b}_{ij}(t) \frac{p^i}{p^j}, \text{ if } p \in R_{++}^l,$$

$$\mathbf{x}^j(t, \mathbf{b}(t), p) = \mathbf{w}^j(t), \text{ otherwise,}$$

$j = 1, \ldots, l$, for each $t \in T$.

According to this rule, given a strategy selection $\mathbf{b}$ and the function $p(\mathbf{b})$, the traders' final holdings are determined as follows:

$$\mathbf{x}(t) = \mathbf{x}(t, \mathbf{b}(t), p(\mathbf{b})),$$

for each $t \in T$. It is straightforward to show that the assignment corresponding to the final holdings is an allocation.

This reformulation of the Shapley's window model allows us to define the following concept of Cournot-Nash equilibrium for exchange economies with an atomless part (see Codognato and Ghosal (2000)).

Definition 3. *A strategy selection* $\hat{\mathbf{b}}$ *such that* $\bar{\bar{\mathbf{B}}}$ *is irreducible is a Cournot-Nash equilibrium of* Γ *if*

$$u_t(\mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))) \geq u_t(\mathbf{x}(t, \hat{\mathbf{b}} \setminus b(t), p(\hat{\mathbf{b}} \setminus b(t)))),$$

for each $b \in \mathbf{B}(t)$ *and for each* $t \in T$.

In what follows, we shall make use of a function $\mathbf{b}^0 : T_0 \to R_+^{l^2}$ such that $\mathbf{b}_{ij}^0(t) = \mathbf{w}^i(t)\boldsymbol{\alpha}^j(t)$, $i, j = 1, \ldots, l$, for each $t \in T_0$. $\mathbf{b}^0(t) \in \mathbf{B}(t)$, for each $t \in T_0$, as $\mathbf{w}^i(t)\boldsymbol{\alpha}^j(t) \geq 0$, $i, j = 1, \ldots, l$, and $\sum_{j=1}^l \mathbf{w}^i(t)\boldsymbol{\alpha}^j(t) = \mathbf{w}^i(t)$, $i = 1, \ldots, l$, for each $t \in T_0$. The following proposition shows that the function $\mathbf{b}^0$ is integrable.

Proposition 2. *Under Assumptions 1, 2, and 3, the function* $\mathbf{b}^0$ *is integrable.*

Proof. $\mathbf{b}^0$ is measurable as the restriction of $\mathbf{w}$ to T_0 is measurable and we know, from the proof of Proposition 1, that $\boldsymbol{\alpha}$ is measurable. Then, $\mathbf{b}^0$ is integrable as $\mathbf{b}_{ij}^0(t) \leq \mathbf{w}^i(t)$, $i, j = 1, \ldots, l$, for each $t \in T_0$. ∎

We now define a game, which we call Γ_1, where only the atoms act strategically, taking $\mathbf{b}^0$ as given. The game Γ_1 can be characterized, *mutatis mutandis*, as Γ. Let $\mathbf{b}^1 : T_1 \to R_+^{l^2}$ be a function such that $\mathbf{b}^1(t) \in \mathbf{B}(t)$, for each $t \in T_1$. $\mathbf{b}^1$ is integrable as $\sum_{t \in T_1} \int_t \mathbf{b}^1(t) \, d\mu \leq \sum_{t \in T_1} \int_t \mathbf{w}(t) \, d\mu = \int_{T_1} \mathbf{w}(t) \, d\mu < \infty$. Then, $\mathbf{b}^1$ is a strategy selection of Γ_1. Given a strategy selection $\mathbf{b}^1$ of Γ_1, let $\mathbf{b}^{10} : T \to R_+^{l^2}$ be a function such that $\mathbf{b}^{10}(t) = \mathbf{b}^1(t)$, for each $t \in T_1$, and $\mathbf{b}^{10}(t) = \mathbf{b}^0(t)$, for each $t \in T_0$. Then, $\mathbf{b}^{10}$ is a strategy selection of Γ as $\int_{T_1} \mathbf{b}^1(t) \, d\mu + \int_{T_0} \mathbf{b}^0(t) \, d\mu \leq \int_{T_1} \mathbf{w}(t) \, d\mu + \int_{T_0} \mathbf{w}(t) \, d\mu = \int_T \mathbf{w}(t) \, d\mu < \infty$. Consider an atom $\tau \in T_1$. Given a strategy selection $\mathbf{b}^{10}$, consider a vector $\bar{b} \in \mathbf{B}(\tau)$. Suppose that $\bar{b}_{ii} \neq \mathbf{b}_{ii}^{10}(\tau)$, for at least a pair (i, i), and $\bar{b}_{ij} = \mathbf{b}_{ij}^{10}(\tau)$, for the remaining pairs (i, j). Then, it is straightforward to verify that $p(\mathbf{b}^{10}) = p(\mathbf{b}^{10} \setminus \bar{b}(\tau))$. Therefore, as in Sahi and Yao (1989), we can assume, for convenience, that, given a strategy selection $\mathbf{b}^{10}$, $\sum_{j=1}^l \mathbf{b}_{ij}^{10}(t) = \mathbf{w}^i(t)$, $i = 1, \ldots, l$, for each $t \in T_1$. Then, given

a strategy selection $\mathbf{b}^{10}$, the corresponding aggregate matrix $\bar{\mathbf{B}}^{10}$ is row-stochastic. Moreover, $\bar{\mathbf{B}}^{10}$ is irreducible as $\int_{T_0} \mathbf{w}(t)\,d\mu \gg 0$ and $\boldsymbol{\alpha}(t) \gg 0$, for each $t \in T_0$.

We can now provide the definition of a Cournot-Nash equilibrium of Γ_1.

Definition 4. *A strategy selection $\hat{\mathbf{b}}^1$ is a Cournot-Nash equilibrium of Γ_1 if*

$$u_t(\mathbf{x}(t, \hat{\mathbf{b}}^1(t), p(\hat{\mathbf{b}}^{10})) \geq u_t(\mathbf{x}(t, \hat{\mathbf{b}}^1 \setminus b(t), p(\hat{\mathbf{b}}^{10} \setminus b(t)))),$$

for each $b \in \mathbf{B}(t)$ and for each $t \in T_1$.

The following argument shows that Lemmas 3 and 4 in Sahi and Yao (1989) still hold in our framework. Consider an atom $\tau \in T_1$. Given a strategy selection $\mathbf{b}^{10}$, let $\mathbf{D}$ be a matrix such that $\mathbf{d}_{ij} = \bar{\mathbf{b}}_{ij}^{10} - \mathbf{b}_{ij}^{10}(\tau)\mu(\tau)$, $i, j = 1, \ldots, l$. Then, from (1), we have

$$\sum_{i=1}^l p^i(\mathbf{b}^{10})(\mathbf{d}_{ij} + \mathbf{b}_{ij}^{10}(\tau)\mu(\tau)) = p^j(\mathbf{b}^{10})(\sum_{i=1}^l (\mathbf{d}_{ji} + \mathbf{b}_{ji}^{10}(\tau)\mu(\tau)), \ j = 1, \ldots, l,$$

from which we obtain

$$-\sum_{i=1}^l \mathbf{b}_{ji}^{10}(\tau) + \sum_{i=1}^l \mathbf{b}_{ij}^{10}(\tau)\frac{p^i(\mathbf{b}^{10})}{p^j(\mathbf{b}^{10})} = \frac{\sum_{i=1}^l \mathbf{d}_{ji}}{\mu(\tau)} - \frac{\sum_{i=1}^l \mathbf{d}_{ij}}{\mu(\tau)}\frac{p^i(\mathbf{b}^{10})}{p^j(\mathbf{b}^{10})}, \ j = 1, \ldots, l.$$

Then,

$$\mathbf{x}^j(\tau, \mathbf{b}^{10}(\tau), p(\mathbf{b}^{10})) = \mathbf{w}^j(\tau) + \frac{\sum_{i=1}^l \mathbf{d}_{ji}}{\mu(\tau)} - \frac{\sum_{i=1}^l \mathbf{d}_{ij}}{\mu(\tau)}\frac{p^i(\mathbf{b}^{10})}{p^j(\mathbf{b}^{10})}, \ j = 1, \ldots, l,$$

from which we obtain

$$(\mu(\tau))\mathbf{x}^j(\tau, \mathbf{b}^{10}(\tau), p(\mathbf{b}^{10})) = 1 - \frac{\sum_{i=1}^l \mathbf{d}_{ij}p^i(\mathbf{b}^{10})}{p^j(\mathbf{b}^{10})}, \ j = 1, \ldots, l. \tag{2}$$

It is possible to show, but we omit the details, that Lemmas 3 and 4 in Sahi and Yao (1989) still hold when their matrices $\mathbf{C}$ and $\mathbf{A}$ are replaced, respectively, with $\mathbf{D}$ and $\bar{\mathbf{B}}^{10}$, and their Equation (14) is replaced with (2).

We can now prove the existence of a Cournot-Nash equilibrium of Γ.

Theorem 1. *Under Assumptions 1, 2, and 3, there exists a Cournot-Nash equilibrium of Γ, $\hat{\mathbf{b}}$.*

Proof. We shall consider the case where T_1 contains countably infinite atoms as the argument we use for this case holds, *a fortiori*, when it contains

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a finite number of atoms. Let $\mathbf{\Phi} : \prod_{t \in T_1} \mathbf{B}(t) \to \prod_{t \in T_1} \mathbf{B}(t)$ be a correspondence such that $\mathbf{\Phi}(\mathbf{b}^1) = \{\mathbf{b}^1 \in \prod_{t \in T_1} \mathbf{B}(t) : \mathbf{b}^1(t) \in \mathbf{\Phi}_t(\mathbf{b}^1)$, for each $t \in T_1\}$ where, for each $t \in T_1$, the correspondence $\mathbf{\Phi}_t : \prod_{t \in T_1} \mathbf{B}(t) \to \mathbf{B}(t)$ is such that $\mathbf{\Phi}_t(\mathbf{b}^1) = \text{argmax}\{u_t(\mathbf{x}(t, \mathbf{b}^1 \setminus b(t), p(\mathbf{b}^{10} \setminus b(t)))) : b \in \mathbf{B}(t)\}$. $\prod_{t \in T_1} R_+^{l^2}$ is a locally convex Hausdorff space as it is a metric space. $\prod_{t \in T_1} \mathbf{B}(t)$ is a nonempty, convex, and compact subset of $\prod_{t \in T_1} R_+^{l^2}$ as $\mathbf{B}(t)$ is nonempty, convex, and compact, for each $t \in T_1$. Consider a trader $\tau \in T_1$. For each $\mathbf{b}^1 \in \prod_{t \in T_1} \mathbf{B}(t)$, $\mathbf{\Phi}_\tau(\mathbf{b}^1)$ is nonempty, convex, and closed, as Lemma 4 in Sahi and Yao (1989) holds in our framework. Moreover, $\mathbf{\Phi}_\tau$ is upper hemicontinuous by the Berge Maximum Theorem (see Theorem 17.31 in Aliprantis and Border (2006), p. 570). Then, $\mathbf{\Phi}_\tau$ has a closed graph, by the Closed Graph Theorem (see Theorem 17.11 in Aliprantis and Border (2006), p. 561) as $\prod_{t \in T_1} \mathbf{B}(t)$ is compact and $\mathbf{\Phi}_\tau$ is upper hemicontinuous and closed-valued. But then, the correspondence $\mathbf{\Phi}$ has nonempty, convex values, and a closed graph. Therefore, by the Kakutani-Fan-Glicksberg Theorem (see Theorem 17.55 in Aliprantis and Border (2006), p. 583) there exists a fixed point $\hat{\mathbf{b}}^1$ of $\mathbf{\Phi}$, which is a Cournot-Nash equilibrium $\hat{\mathbf{b}}^1$ of Γ_1. Let $\hat{\mathbf{b}}$ be a strategy selection of Γ such that $\hat{\mathbf{b}}(t) = \hat{\mathbf{b}}^{10}(t)$, for each $t \in T$. $\bar{\bar{\mathbf{B}}}$ is irreducible as $\bar{\bar{\mathbf{B}}}^{10}$ is irreducible. Consider a trader $\tau \in T_1$. Then, $u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}}(\tau), p(\hat{\mathbf{b}}))) \geq u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}} \setminus b(\tau), p(\hat{\mathbf{b}} \setminus b(\tau))))$, for each $b \in \mathbf{B}(\tau)$, as $\hat{\mathbf{b}}^1$ is a Cournot-Nash equilibrium of Γ_1. Consider a trader $\tau \in T_0$. $\mathbf{x}(\tau, \hat{\mathbf{b}}(\tau), p(\hat{\mathbf{b}})) \in \mathbf{X}_{p(\hat{\mathbf{b}})}(\tau)$ as $\mathbf{x}^j(\tau, \hat{\mathbf{b}}(\tau), p(\hat{\mathbf{b}})) = \frac{\boldsymbol{\alpha}^j(\tau) \sum_{i=1}^l p^i(\hat{\mathbf{b}}) \mathbf{w}^i(\tau)}{p^j(\hat{\mathbf{b}})}$, $j = 1, \ldots, l$. Suppose that there exists $\bar{b} \in \mathbf{B}(\tau)$ such that $u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}} \setminus \bar{b}(\tau), p(\hat{\mathbf{b}} \setminus \bar{b}(\tau)))) > u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}}(\tau), p(\hat{\mathbf{b}})))$. It is immediate to verify that $p(\hat{\mathbf{b}} \setminus \bar{b}(\tau)) = p(\hat{\mathbf{b}})$. Let $\bar{x} = \mathbf{x}(\tau, \hat{\mathbf{b}} \setminus \bar{b}(\tau), p(\hat{\mathbf{b}}))$. Then, it is straightforward so show that $\bar{x} \in \mathbf{\Delta}_{p(\hat{\mathbf{b}})}(\tau)$. But then, $u_\tau(\bar{x}) > u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}}(\tau), p(\hat{\mathbf{b}})))$ and $\bar{x} \in \mathbf{\Delta}_{p(\hat{\mathbf{b}})}(\tau)$, a contradiction. Therefore, $u_t(\mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))) \geq u_t(\mathbf{x}(t, \hat{\mathbf{b}} \setminus b(t), p(\hat{\mathbf{b}} \setminus b(t))))$, for each $b \in \mathbf{B}(t)$ and for each $t \in T$. Hence, $\hat{\mathbf{b}}$ is a Cournot-Nash equilibrium of Γ. ∎

A Cournot-Nash equilibrium $\hat{\mathbf{b}}$ of Γ is said to be a Cobb-Douglas-Cournot-Nash equilibrium of Γ if $\hat{\mathbf{b}}(t) = \mathbf{b}^0$, for each $t \in T_0$. The following Corollary is a straightforward consequence of Theorem 1.

Corollary 1. *Under Assumptions 1, 2, and 3, there exists a Cobb-Douglas-Cournot-Nash equilibrium of Γ, $\hat{\mathbf{b}}$.*

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4   Cournot-Walras equilibrium

In this section, we describe the concept of Cournot-Walras equilibrium proposed by Busetto et al. (2008). The atomless part has Walrasian demands represented by the function $\mathbf{x}^0(\cdot, p) : T_0 \to R_+^l$, defined in Section 2. Consider now the atoms' strategies. Let $e \in R^{l^2}$ be a vector such that $e = (e_{11}, e_{12}, \ldots, e_{ll-1}, e_{ll})$. A strategy correspondence is a correspondence $\mathbf{E} : T_1 \to \mathcal{P}(R^{l^2})$ such that, for each $t \in T_1$, $\mathbf{E}(t) = \{e \in R^{l^2} : e_{ij} \geq 0, i, j = 1, \ldots, l; \sum_{j=1}^l e_{ij} \leq \mathbf{w}^i(t), i = 1, \ldots, l\}$. A strategy selection is an integrable function $\mathbf{e} : T_1 \to R^{l^2}$ such that, for each $t \in T_1$, $\mathbf{e}(t) \in \mathbf{E}(t)$. For each $t \in T_1$, $\mathbf{e}_{ij}(t)$, $i, j = 1, \ldots, l$, represents the amount of commodity i that trader t offers in exchange for commodity j. We denote by $\mathbf{e} \setminus e(t)$ a strategy selection obtained by replacing $\mathbf{e}(t)$ in $\mathbf{e}$ with $e \in \mathbf{E}(t)$. With a slight abuse of notation, $\mathbf{e} \setminus e(t)$ will also denote the value of the strategy selection $\mathbf{e} \setminus e(t)$ at t. Given a strategy selection $\mathbf{e}$, consider the following equation:

$$\int_{T_0} \mathbf{x}^{0j}(t, p)\, d\mu + \sum_{i=1}^l \int_{T_1} \mathbf{e}_{ij}(t)\, d\mu \frac{p^i}{p^j} = \int_{T_0} \mathbf{w}^j(t)\, d\mu + \sum_{i=1}^l \int_{T_1} \mathbf{e}_{ji}(t)\, d\mu, \quad (3)$$

$j = 1, \ldots, l$. The following proposition shows that there exists a unique, up to a scalar multiple, price vector $p \in R_{++}^l$ which satisfies Equation (3).

Proposition 3. *Under Assumptions 1, 2, and 3, for each strategy selection* $\mathbf{e}$*, that there exists a unique, up to a scalar multiple, price vector* $p \in R_{++}^l$ *which satisfies Equation (3).*

Proof. Consider a strategy selection $\mathbf{e}$. Let $\mathbf{e}^{10} : T \to R_+^{l^2}$ be a function such that $\mathbf{e}^{10}(t) = \mathbf{e}(t)$, for each $t \in T_1$, and $\mathbf{e}^{10}(t) = \mathbf{b}^0(t)$, for each $t \in T_0$. Then, $\mathbf{e}^{10}$ is integrable by the same argument used for the function $\mathbf{b}^{10}$. Define the aggregate matrix $\bar{\mathbf{E}}^{10} = (\int_T \mathbf{e}_{ij}^{10}(t)\, d\mu)$. $\bar{\mathbf{E}}^{10}$ is irreducible by the same argument used for the matrix $\bar{\mathbf{B}}^{10}$. (3) can be written as

$$\sum_{i=1}^l p^i \left(\int_{T_0} \mathbf{w}^i(t) \boldsymbol{\alpha}^j(t)\, d\mu + \int_{T_1} \mathbf{e}_{ij}(t)\, d\mu \right)$$

$$= p^j \left(\sum_{i=1}^l \left(\int_{T_0} \mathbf{w}^j(t) \boldsymbol{\alpha}^i(t)\, d\mu + \int_{T_1} \mathbf{e}_{ji}(t)\, d\mu \right) \right), \quad j = 1, \ldots, l.$$

Then, (3) can be rewritten as

$$\sum_{i=1}^{l} p^i \bar{\mathbf{e}}_{ij}^{10} = p^j (\sum_{i=1}^{l} \bar{\mathbf{e}}_{ji}^{10}), \ j = 1, \ldots, l. \tag{4}$$

By Lemma 1 in Sahi and Yao (1989), there is a unique, up to a scalar multiple, price vector $p \in R_{++}^l$ satisfying (4) as $\bar{\mathbf{E}}^{10}$ is irreducible. Hence, there exists a unique, up to a scalar multiple, price vector $p \in R_{++}^l$ which satisfies Equation (3). $\blacksquare$

We denote by $p(\mathbf{e})$ a function which associates, with each strategy selection $\mathbf{e}$, the unique, up to a scalar multiple, price vector p satisfying (3). It is straightforward to verify that $p(\mathbf{e}') = p(\mathbf{e}'')$ if $\int_{T_1} \mathbf{e}'(t) \, d\mu = \int_{T_1} \mathbf{e}''(t) \, d\mu$. For each strategy selection $\mathbf{e}$, let $\mathbf{x}^1(\cdot, \mathbf{e}(\cdot), p(\mathbf{e})) : T_1 \to R_+^l$ denote a function such that

$$\mathbf{x}^{1j}(t, \mathbf{e}(t), p(\mathbf{e})) = \mathbf{w}^j(t) - \sum_{i=1}^{l} \mathbf{e}_{ji}(t) + \sum_{i=1}^{l} \mathbf{e}_{ij}(t) \frac{p^i(\mathbf{e})}{p^j(\mathbf{e})}, \tag{5}$$

$j = 1, \ldots, l$, for each $t \in T_1$. Given a strategy selection $\mathbf{e}$, taking into account the structure of the traders' measure space, Proposition 3, and Equation (3), it is straightforward to show that the function $\mathbf{x}(t)$ such that $\mathbf{x}(t) = \mathbf{x}^1(t, \mathbf{e}(t), p(\mathbf{e}))$, for all $t \in T_1$, and $\mathbf{x}(t) = \mathbf{x}^0(t, p(\mathbf{e}))$, for all $t \in T_0$, is an allocation.

At this stage, we are able to define the concept of Cournot-Walras equilibrium.

Definition 5. *A pair $(\tilde{\mathbf{e}}, \tilde{\mathbf{x}})$, consisting of a strategy selection $\tilde{\mathbf{e}}$ and an allocation $\tilde{\mathbf{x}}$ such that $\tilde{\mathbf{x}}(t) = \mathbf{x}^1(t, \tilde{\mathbf{e}}(t), p(\tilde{\mathbf{e}}))$, for each $t \in T_1$, and $\tilde{\mathbf{x}}(t) = \mathbf{x}^0(t, p(\tilde{\mathbf{e}}))$, for each $t \in T_0$, is a Cournot-Walras equilibrium of $\mathcal{E}$ if*

$$u_t(\mathbf{x}^1(t, \tilde{\mathbf{e}}(t), p(\tilde{\mathbf{e}}))) \geq u_t(\mathbf{x}^1(t, \tilde{\mathbf{e}} \setminus e(t), p(\tilde{\mathbf{e}} \setminus e(t)))),$$

for each $e \in \mathbf{E}(t)$ and for each $t \in T_1$.

5   Cournot-Nash and Cournot-Walras equilibrium

The following theorem shows the equivalence between the set of the Cobb-Douglas-Cournot-Nash equilibrium allocations and the set of the Cournot-Walras equilibrium allocations.

Theorem 2. *Under Assumptions 1, 2, and 3, (i) if $\hat{\mathbf{b}}$ is a Cobb-Douglas-Cournot-Nash equilibrium of Γ, then there is a strategy selection $\hat{\mathbf{e}}$ such that the pair $(\hat{\mathbf{e}}, \hat{\mathbf{x}})$, where $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}})) = \mathbf{x}^1(t, \hat{\mathbf{e}}(t), p^{10}(\hat{\mathbf{e}}))$, for each $t \in T_1$, and $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}})) = \mathbf{x}^0(t, p^{10}(\hat{\mathbf{e}}))$, for each $t \in T_0$, is a Cournot-Walras equilibrium of $\mathcal{E}$; (ii) if $(\tilde{\mathbf{e}}, \tilde{\mathbf{x}})$ is a Cournot-Walras equilibrium of $\mathcal{E}$, then there is a Cobb-Douglas-Cournot-Nash equilibrium $\tilde{\mathbf{b}}$ of Γ such that $\tilde{\mathbf{x}}(t) = \mathbf{x}(t, \tilde{\mathbf{b}}(t), p(\tilde{\mathbf{b}}))$, for each $t \in T$.*

Proof. (i) Let $\hat{\mathbf{b}}$ be a Cobb-Douglas-Cournot-Nash equilibrium of Γ. Let $\hat{\mathbf{e}}$ be a strategy selection such that $\hat{\mathbf{e}}(t) = \hat{\mathbf{b}}(t)$, for each $t \in T_1$. Then, $p(\hat{\mathbf{e}}) = p(\hat{\mathbf{b}})$ as $\bar{\bar{\mathbf{E}}}^{10} = \bar{\bar{\mathbf{B}}}$ and $p(\hat{\mathbf{b}})$ satisfies Equation (1). But then, it is straightforward to verify that $\mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}})) = \mathbf{x}^1(t, \hat{\mathbf{e}}(t), p(\hat{\mathbf{e}}))$, for each $t \in T_1$, and $\mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}})) = \mathbf{x}^0(t, p(\hat{\mathbf{e}}))$, each $t \in T_0$. Suppose that there is a trader $\tau \in T_1$ and a strategy $\bar{e} \in \mathbf{E}(\tau)$ such that $u_\tau(\mathbf{x}^1(\tau, \hat{\mathbf{e}} \setminus \bar{e}(\tau), p(\hat{\mathbf{e}} \setminus \bar{e}(\tau)))) > u_\tau(\mathbf{x}^1(\tau, \hat{\mathbf{e}}(\tau), p(\hat{\mathbf{e}})))$. Then, $u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}} \setminus \bar{e}(\tau), p(\hat{\mathbf{b}} \setminus \bar{e}(\tau)))) = u_\tau(\mathbf{x}^1(\tau, \hat{\mathbf{e}} \setminus \bar{e}(\tau), p(\hat{\mathbf{e}} \setminus \bar{e}(\tau)))) > u_\tau(\mathbf{x}^1(\tau, \hat{\mathbf{e}}(\tau), p(\hat{\mathbf{e}}))) = u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}}(\tau), p(\hat{\mathbf{b}})))$ as $p^{10}(\hat{\mathbf{e}} \setminus \bar{e}(\tau)) = p(\hat{\mathbf{b}} \setminus \bar{e}(\tau))$, a contradiction. Therefore, $u_t(\mathbf{x}^1(t, \hat{\mathbf{e}}(t), p(\hat{\mathbf{e}}))) \geq u_t(\mathbf{x}^1(t, \hat{\mathbf{e}} \setminus e(t), p(\hat{\mathbf{e}} \setminus e(t))))$, for each $e \in \mathbf{E}(t)$ and for each $t \in T_1$. Hence, the pair $(\hat{\mathbf{e}}, \hat{\mathbf{x}})$, where $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}})) = \mathbf{x}^1(t, \hat{\mathbf{e}}(t), p^{10}(\hat{\mathbf{e}}))$, for each $t \in T_1$, and $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}})) = \mathbf{x}^0(t, p^{10}(\hat{\mathbf{e}}))$, for each $t \in T_0$, is a Cournot-Walras equilibrium of $\mathcal{E}$. (ii) Let $(\tilde{\mathbf{e}}, \tilde{\mathbf{x}})$ be a Cournot-Walras equilibrium of $\mathcal{E}$. Let $\tilde{\mathbf{b}}$ be a strategy selection such that $\tilde{\mathbf{b}}(t) = \tilde{\mathbf{e}}^{10}(t)$, for each $t \in T$. Then, $\bar{\bar{\mathbf{B}}}$ is irreducible and $p(\tilde{\mathbf{b}}) = p(\tilde{\mathbf{e}})$ as $\bar{\bar{\mathbf{B}}} = \bar{\bar{\mathbf{E}}}^{10}$ and $p(\tilde{\mathbf{e}})$ satisfies Equation (4). But then, it is straightforward to verify that $\tilde{\mathbf{x}}(t) = \mathbf{x}(t, \tilde{\mathbf{b}}(t), p(\tilde{\mathbf{b}}))$, for each $t \in T$. Suppose that there is a trader $\tau \in T_1$ and a strategy $\bar{b} \in \mathbf{B}(\tau)$ such that $u_\tau(\mathbf{x}(\tau, \tilde{\mathbf{b}} \setminus \bar{b}(\tau), p(\tilde{\mathbf{b}} \setminus \bar{b}(\tau)))) > u_\tau(\mathbf{x}(\tau, \tilde{\mathbf{b}}(\tau), p(\tilde{\mathbf{b}})))$. Then, $u_\tau(\mathbf{x}^1(\tau, \tilde{\mathbf{e}} \setminus \bar{b}(\tau), p(\tilde{\mathbf{e}} \setminus \bar{b}(\tau)))) = u_\tau(\mathbf{x}(\tau, \tilde{\mathbf{b}} \setminus \bar{b}(\tau), p(\tilde{\mathbf{b}} \setminus \bar{b}(\tau)))) > u_\tau(\mathbf{x}(\tau, \tilde{\mathbf{b}}(\tau), p(\tilde{\mathbf{b}}))) = u_\tau(\mathbf{x}^1(\tau, \tilde{\mathbf{e}}(\tau), p(\tilde{\mathbf{e}})))$, as $p(\tilde{\mathbf{b}} \setminus \bar{b}(\tau)) = p(\tilde{\mathbf{e}} \setminus \bar{b}(\tau))$, a contradiction. Therefore, $u_t(\mathbf{x}(t, \tilde{\mathbf{b}}(t), p(\tilde{\mathbf{b}})) \geq u_t(\mathbf{x}(t, \tilde{\mathbf{b}} \setminus b(t), p(\tilde{\mathbf{b}} \setminus b(t))))$, for each $b \in \mathbf{B}(t)$ and for each $t \in T_1$. Moreover, $u_t(\mathbf{x}(t, \tilde{\mathbf{b}}(t), p(\tilde{\mathbf{b}})) \geq u_t(\mathbf{x}(t, \tilde{\mathbf{b}} \setminus b(t), p(\tilde{\mathbf{b}} \setminus b(t))))$, for each $b \in \mathbf{B}(t)$ and for each $t \in T_0$, by the same argument used in the proof of Theorem 1. Hence, $\tilde{\mathbf{b}}$ is a Cobb-Douglas-Cournot-Nash equilibrium of Γ. ∎

The following corollary is a straightforward consequence of Theorem 2.

Corollary 2. *Under Assumptions 1, 2, and 3, there exists a Cournot-Walras equilibrium of $\mathcal{E}$, $(\tilde{\mathbf{e}}, \tilde{\mathbf{x}})$.*

6 Cournot-Nash and Walras equilibrium

As in Busetto et al. (2012), we consider the replication à la Cournot of $\mathcal{E}$ which, by analogy with the replication proposed by Cournot (1838) in a partial equilibrium framework, consists in replicating only the atoms of $\mathcal{E}$, while making them asymptotically negligible. Let $\mathcal{E}^n$ be an exchange economy characterized as in Section 2 where each atom is replicated n times. For each $t \in T_1$, let tr denote r-th element of the n-fold replication of t. We assume that $\mathbf{w}(tr) = \mathbf{w}(ts) = \mathbf{w}(t)$, $u_{tr}(\cdot) = u_{ts}(\cdot) = u_t(\cdot)$, $r, s = 1, \ldots, n$, $\mu(tr) = \frac{\mu(t)}{n}$, $r = 1, \ldots, n$, for each $t \in T_1$.

The strategic market game Γ^n associated with $\mathcal{E}^n$ can then be characterized, *mutatis mutandis*, as in Section 2. A strategy selection $\mathbf{b}$ of Γ^n is atom-type-symmetric if $\mathbf{b}^n(tr) = \mathbf{b}^n(ts)$, $r, s = 1, \ldots, n$, for each $t \in T_1$. We can now provide the definition of an atom-type-symmetric Cournot-Nash equilibrium of Γ^n.

Definition 6. *A strategy selection $\hat{\mathbf{b}}$ such that $\bar{\bar{\mathbf{B}}}$ is irreducible is an atom-type-symmetric Cournot-Nash equilibrium of Γ^n if $\hat{\mathbf{b}}$ is atom-type-symmetric and*

$$u_{tr}(\mathbf{x}(tr, \hat{\mathbf{b}}(tr), p(\hat{\mathbf{b}}))) \geq u_{tr}(\mathbf{x}(tr, \hat{\mathbf{b}} \setminus b(tr), p(\hat{\mathbf{b}} \setminus b(tr)))),$$

for all $b \in \mathbf{B}(tr)$, $r = 1, \ldots, n$, and for each $t \in T_1$;

$$u_t(\mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))) \geq u_t(\mathbf{x}(t, \hat{\mathbf{b}} \setminus b(t), p(\hat{\mathbf{b}} \setminus b(t)))),$$

for all $b \in \mathbf{B}(t)$ and for each $t \in T_0$.

We now define a game, which we call Γ_1^n, where only the atoms act strategically, taking $\mathbf{b}^0$ as given. The game Γ_1^n can be characterized, *mutatis mutandis*, as Γ_1. Moreover, $\mathbf{b}^1$ and $\mathbf{b}^{10}(t)$ can be defined for Γ_1^n, *mutatis mutandis*, as they were defined for Γ_1. Then, the notion of a Cournot-Nash equilibrium $\hat{\mathbf{b}}^1$ of Γ_1^n can be provided as in Definition 4 and the existence of such an equilibrium can be proved using the same argument as in Theorem 1. A strategy selection $\mathbf{b}^1$ of Γ_1^n is atom-type-symmetric if $\mathbf{b}^n(tr) = \mathbf{b}^n(ts)$, $r, s = 1, \ldots, n$, for each $t \in T_1$.

The following theorem shows the existence of an atom-type-symmetric Cournot-Nash equilibrium $\hat{\mathbf{b}}^1$ of Γ^n.

Theorem 3. *Under Assumptions 1, 2, and 3, there exists an atom-type-symmetric Cournot-Nash equilibrium of Γ^n, $\hat{\mathbf{b}}$.*

Proof. We shall consider the case where T_1 contains countably infinite atoms as the argument we use for this case holds, *a fortiori*, when it contains

a finite number of atoms. Let B^* denote the subset of $\prod_{t\in T_1}\prod_{r=1}^n \mathbf{B}(tr)$ which contains all the atom-type-symmetric strategy selections of Γ_1^n. B^* is convex and compact as B^* is closed, $B^* \subset \prod_{t\in T_1}\prod_{r=1}^n \mathbf{B}(tr)$, and $\prod_{t\in T_1}\prod_{r=1}^n \mathbf{B}(tr)$ is convex and compact. Let $\mathbf{\Phi}^n : \prod_{t\in T_1}\prod_{r=1}^n \mathbf{B}(tr) \to \prod_{t\in T_1}\prod_{r=1}^n \mathbf{B}(tr)$ be a correspondence defined as in the proof of Theorem 1. $\mathbf{\Phi}^n$ has nonempty, convex values, and a closed graph, by the same argument of the proof of Theorem 1. Moreover, let $\mathbf{\Phi}^{n*} : B^* \to B^*$ be a correspondence such that $\mathbf{\Phi}^{n*}(\mathbf{b}^1) = \mathbf{\Phi}^n(\mathbf{b}^1) \cap B^*$, for each $\mathbf{b}^1 \in B^*$. For each $\mathbf{b}^1 \in B^*$ and for each $t \in T_1$, there exists $\bar{b} \in \mathbf{B}(t)$ such that $\bar{b} \in \mathbf{\Phi}_{tr}^{n*}(\mathbf{b})$, $r = 1,\dots,n$, as $\mathbf{b}^1$ is an atom-type-symmetric strategy profile. Then, $\mathbf{\Phi}^{n*}$ is nonempty. Moreover, $\mathbf{\Phi}^{n*}$ has convex values as, for each $\mathbf{b}^1 \in B^*$, $\mathbf{\Phi}^{n*}(\mathbf{b}^1) = \mathbf{\Phi}^n(\mathbf{b}^1) \cap B^*$, $\mathbf{\Phi}^n(\mathbf{b}^1)$ is convex, and B^* is convex. Finally, $\mathbf{\Phi}^{n*}$ has a closed graph as it is the intersection of the correspondence $\mathbf{\Phi}^n$ and the continuous correspondence which assigns, to each strategy selection $\mathbf{b}^1 \in B^*$, the compact set B^* which, by the Closed Graph Theorem, has a closed graph (see Theorem 17.25 in Aliprantis and Border (2006), p. 567). Therefore, by the Kakutani-Fan-Glicksberg Theorem, there exists a fixed point $\hat{\mathbf{b}}^1$ of $\mathbf{\Phi}^{n*}$ which is an atom-type-symmetric Cournot-Nash equilibrium of Γ^1. Let $\hat{\mathbf{b}}$ be a strategy selection of Γ^n such that $\hat{\mathbf{b}}(t) = \hat{\mathbf{b}}^{10}(t)$, for each $t \in T$. Hence, by the same argument used in the proof of Theorem 1, $\hat{\mathbf{b}}$ is an atom-type-symmetric Cournot-Nash equilibrium of Γ^n. ∎

An atom-type-symmetric Cournot-Nash equilibrium $\hat{\mathbf{b}}$ of Γ^n is said to be a Cobb-Douglas-atom-type-symmetric Cournot-Nash equilibrium of Γ^n if $\hat{\mathbf{b}}(t) = \mathbf{b}^0(t)$, for each $t \in T_0$. The following Corollary is a straightforward consequence of Theorem 3.

Corollary 3. *Under Assumptions 1, 2, and 3, there exists a Cobb-Douglas-atom-type-symmetric-Cournot-Nash equilibrium of Γ^n, $\hat{\mathbf{b}}$.*

The following theorem shows that the sequences of Cournot-Nash equilibrium allocations generated by the replication à la Cournot of $\mathcal{E}$ approximate a Walras equilibrium allocation of $\mathcal{E}$.

Theorem 4. *Under Assumptions 1, 2, and 3, let $\{\hat{\mathbf{b}}^n\}$ be a sequence of strategy selections of Γ and let $\{\hat{p}^n\}$ be a sequence of prices such that $\hat{\mathbf{b}}^n(t) = \hat{\mathbf{b}}^{\Gamma^n}(tr)$, $r = 1,\dots,n$, for each $t \in T_1$, $\hat{\mathbf{b}}^n(t) = \hat{\mathbf{b}}^{\Gamma^n}(t)$, for each $t \in T_0$, $\sum_{i=1}^l \hat{p}^{in} = 1$, and $\hat{p}^n = p(\hat{\mathbf{b}}^{\Gamma^n})$, where $\hat{\mathbf{b}}^{\Gamma^n}$ is a Cobb-Douglas-atom-type-symmetric Cournot-Nash equilibrium of Γ^n, for $n = 1,2,\dots$. Then, (i) there exists a subsequence $\{\hat{\mathbf{b}}^{k_n}\}$ of the sequence $\{\hat{\mathbf{b}}^n\}$, a subsequence $\{\hat{p}^{k_n}\}$ of the sequence $\{\hat{p}^n\}$, a strategy selection $\hat{\mathbf{b}}$ of Γ, and a price vector $\hat{p}$,*

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with $p \gg 0$, such that $\hat{\mathbf{b}}(t)$ is the limit of the sequence $\{\hat{\mathbf{b}}^{k_n}(t)\}$, for each $t \in T$, and the sequence $\{\hat{p}^{k_n}\}$ converges to $\hat{p}$; (ii) $\hat{\mathbf{x}}(t)$ is the limit of the sequence $\{\hat{\mathbf{x}}^{k_n}(t)\}$, for each $t \in T$, where $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), \hat{p})$ for each $t \in T$, $\hat{\mathbf{x}}^{k_n}(t) = \mathbf{x}(t, \hat{\mathbf{b}}^{k_n}(t), \hat{p}^{k_n})$, for each $t \in T$, and for $n = 1, 2, \ldots$; (iii) The pair $(\hat{p}, \hat{\mathbf{x}})$ is a Walras equilibrium of $\mathcal{E}$.

Proof (i) Let $\{\hat{\mathbf{b}}^n\}$ be a sequence of strategy selections of Γ and let $\{\hat{p}^n\}$ be a sequence of prices such that $\hat{\mathbf{b}}^n(t) = \hat{\mathbf{b}}^{\Gamma^n}(tr)$, $r = 1, \ldots, n$, for each $t \in T_1$, $\hat{\mathbf{b}}^n(t) = \hat{\mathbf{b}}^{\Gamma^n}(t)$, for each $t \in T_0$, $\sum_{i=1}^{l} \hat{p}^{in} = 1$, and $\hat{p}^n = p(\hat{\mathbf{b}}^{\Gamma^n})$, where $\hat{\mathbf{b}}^{\Gamma^n}$ is a Cobb-Douglas-atom-type-symmetric Cournot-Nash equilibrium of Γ^n, for $n = 1, 2, \ldots$. Let $\hat{\mathbf{b}}^{1n}$ denote the restriction of $\hat{\mathbf{b}}^n$ to T_1, for $n = 1, 2, \ldots$. The fact the sequence $\{\hat{\mathbf{b}}^{1n}(t)\}$ belongs to the compact set $\mathbf{B}(t)$, for each $t \in T_1$, $\hat{\mathbf{b}}^n(t) = \mathbf{b}^0(t)$, for each $t \in T_0$, and the sequence $\{\hat{p}^n\}$ belongs to a compact set P, implies that there is a subsequence $\{\hat{\mathbf{b}}^{1k_n}(t)\}$ of the sequence $\{\hat{\mathbf{b}}^{1n}(t)\}$ which converges to an element $\hat{\mathbf{b}}^1(t)$ of the set $\mathbf{B}(t)$, for each $t \in T_1$, a subsequence $\{\hat{\mathbf{b}}^{k_n}(t)\}$ of the sequence $\{\hat{\mathbf{b}}^n(t)\}$ which converges to $\mathbf{b}^0(t)$, for each $t \in T_0$, and a subsequence $\{\hat{p}^{k_n}\}$ of the sequence $\{\hat{p}^n\}$ which converges to an element $\hat{p}$ of the set P. Let $\hat{\mathbf{b}}(t) = \mathbf{b}^1(t)$, for each $t \in T_1$, and $\hat{\mathbf{b}}(t) = \mathbf{b}^0(t)$, for each $t \in T_0$. Then, by the same argument used in the proof of Theorem 1, $\hat{\mathbf{b}}$ is a strategy selection of Γ and $\hat{\bar{\mathbf{B}}}$ is irreducible. Let $\hat{\mathbf{b}}^1$ denote the restriction of $\hat{\mathbf{b}}$ to T_1. The sequence $\int_{T_1} \hat{\mathbf{b}}^{1k_n}(t)\, d\mu$ converges to $\int_{T_1} \hat{\mathbf{b}}^1(t)\, d\mu$, by the Lebesgue Dominated Convergence Theorem (see Aliprantis and Border (2006) p. 415), as the sequence $\{\hat{\mathbf{b}}^{1k_n}\}$ converges to $\hat{\mathbf{b}}^1$ and $\hat{\mathbf{b}}^{1k_n}(t) \in \mathbf{B}(t)$, for each $t \in T_1$, and for $n = 1, 2, \ldots$. Then, the sequence $\{\int_{t \in T} \hat{\mathbf{b}}^{k_n}(t)\, d\mu\}$ converges to $\int_T \hat{\mathbf{b}}(t)\, d\mu$ as $\int_{t \in T} \hat{\mathbf{b}}^{k_n}(t)\, d\mu = \int_{t \in T_1} \hat{\mathbf{b}}^{k_n}(t)\, d\mu + \int_{t \in T_0} \hat{\mathbf{b}}^{k_n}(t)\, d\mu = \int_{t \in T_1} \hat{\mathbf{b}}^{1k_n}(t)\, d\mu + \int_{t \in T_0} \hat{\mathbf{b}}^0(t)\, d\mu$, for $n = 1, 2, \ldots$, the sequence $\int_{T_1} \hat{\mathbf{b}}^{1k_n}(t)\, d\mu$ converges to $\int_{T_1} \hat{\mathbf{b}}^1(t)\, d\mu$, and $\int_T \hat{\mathbf{b}}(t)\, d\mu = \int_{t \in T_1} \hat{\mathbf{b}}^1(t)\, d\mu + \int_{t \in T_0} \hat{\mathbf{b}}^0(t)\, d\mu$. Therefore, the sequence $\{\hat{\bar{\mathbf{B}}}^{k_n}\}$ converges to $\hat{\bar{\mathbf{B}}}$. Moreover, $\overline{\hat{\mathbf{B}}^{\Gamma^n}} = \overline{\hat{\mathbf{B}}^n}$ as $\overline{\hat{\mathbf{b}}^{\Gamma^n}_{ij}} = \sum_{t \in T_1} \sum_{r=1}^{n} \hat{\mathbf{b}}^{\Gamma^n}_{ij}(tr)\mu(tr) + \int_{t \in T_0} \hat{\mathbf{b}}^{\Gamma^n}_{ij}(t)\, d\mu = \sum_{t \in T_1} n\hat{\mathbf{b}}^n_{ij}(t)\frac{\mu(t)}{n} + \int_{t \in T_0} \hat{\mathbf{b}}^n_{ij}(t)\, d\mu = \sum_{t \in T_1} \hat{\mathbf{b}}^n_{ij}(t)\mu(t) + \int_{t \in T_0} \hat{\mathbf{b}}^n_{ij}(t)\, d\mu = \overline{\hat{\mathbf{b}}^n_{ij}}$, $i, j = 1, \ldots, l$, for $n = 1, 2, \ldots$. Then, $\hat{p}^n = p(\hat{\mathbf{b}}^n)$ as $\hat{p}^n$ and $\hat{\mathbf{b}}^n$ satisfy (1), for $n = 1, 2, \ldots$. But then, by continuity, $\hat{p}$ and $\hat{\mathbf{b}}$ must satisfy (1) as the sequence $\{\hat{p}^{k_n}\}$ converges to $\hat{p}$ and the sequence $\{\hat{\bar{\mathbf{B}}}^{k_n}\}$ converges to $\hat{\bar{\mathbf{B}}}$. Therefore, Lemma 1 in Sahi and Yao (1989) implies that $\hat{p} \gg 0$. (ii) Let $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), \hat{p})$, for each $t \in T$, $\hat{\mathbf{x}}^{k_n}(t) = \mathbf{x}(t, \hat{\mathbf{b}}^{k_n}(t), \hat{p}^{k_n})$, for each $t \in T$, and for $n = 1, 2, \ldots$. Then, $\hat{\mathbf{x}}(t)$ is the limit of the sequence $\{\hat{\mathbf{x}}^{k_n}(t)\}$, for each $t \in T$, as $\hat{\mathbf{b}}(t)$ is the limit

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of the sequence $\{\hat{\mathbf{b}}^{k_n}(t)\}$, for each $t \in T$, and the sequence $\{\hat{p}^{k_n}\}$ converges to $\hat{p}$. (iii) Consider the pair $(\hat{p}, \hat{\mathbf{x}})$. It is straightforward to show that the assignment $\hat{\mathbf{x}}$ is an allocation as $\hat{p}$ and $\hat{\mathbf{b}}$ satisfy (1) and that $\hat{\mathbf{x}}(t) \in \mathbf{\Delta}_{\hat{p}}(t)$, for all $t \in T$. Suppose that $(\hat{p}, \hat{\mathbf{x}})$ is not a Walras equilibrium of $\mathcal{E}$. Then, there exists a trader $\tau \in T$ and a commodity bundle $\bar{x} \in \mathbf{\Delta}_{\hat{p}}(t)$ such that $u_\tau(\bar{x}) > u_\tau(\hat{\mathbf{x}}(\tau))$. By Lemma 5 in Codognato and Ghosal (2000), there exist $\bar{\lambda}^j \geq 0$, $\sum_{j=1}^{l} \bar{\lambda}^j = 1$, such that

$$\bar{x}^j = \bar{\lambda}^j \frac{\sum_{i=1}^{l} \hat{p}^i \mathbf{w}^i(\tau)}{\hat{p}^j}, \ j = 1, \ldots, l.$$

Let $\bar{b}_{ij} = \mathbf{w}^i(\tau)\bar{\lambda}^j$, $i, j = 1, \ldots, l$. Then, it is straightforward to verify that

$$\bar{x}^j = \mathbf{w}^j(\tau) - \sum_{i=1}^{l} \bar{b}_{ji} + \sum_{i=1}^{l} \bar{b}_{ij} \frac{\hat{p}^i}{\hat{p}^j},$$

for each $j = 1, \ldots, l$. Consider first the case where $\tau \in T_1$. Let ρ denote the k_1-th element of the n-fold replication of $\mathcal{E}$ and let $\hat{\mathbf{B}}^{\Gamma k_n} \setminus \bar{b}(\tau\rho)$ denote the aggregate matrix corresponding to the strategy selection $\hat{\mathbf{b}}^{\Gamma k_n} \setminus \bar{b}(\tau\rho)$, where $\bar{b}(\tau\rho) = \bar{b}$, for $n = 1, 2, \ldots$. Let $\Delta\hat{\mathbf{B}}^{\Gamma k_n}$, $\Delta\hat{\mathbf{B}}^{\Gamma k_n} \setminus \bar{b}(\tau\rho)$, and $\Delta\hat{\mathbf{B}}^{k_n}$ denote the matrices of row sums of, respectively, $\hat{\mathbf{B}}^{\Gamma k_n}$, $\hat{\mathbf{B}}^{\Gamma k_n} \setminus \bar{b}(\tau\rho)$, and $\hat{\mathbf{B}}^{k_n}$, for $n = 1, 2, \ldots$. Moreover, let $q^{\Gamma k_n}$, $q^{k_n}_{\tau\rho}$, and q^{k_n} denote the vectors of the cofactors of the first column of, respectively, $\Delta\hat{\mathbf{B}}^{\Gamma k_n} - \hat{\mathbf{B}}^{\Gamma k_n}$, $\Delta\hat{\mathbf{B}}^{\Gamma k_n} \setminus \bar{b}(\tau\rho) - \hat{\mathbf{B}}^{\Gamma k_n} \setminus \bar{b}(\tau\rho)$, and $\Delta\hat{\mathbf{B}}^{k_n} - \hat{\mathbf{B}}^{k_n}$, for $n = 1, 2, \ldots$. Clearly, $q^{\Gamma k_n} = q^{k_n}$ as $\hat{\mathbf{B}}^{\Gamma k_n} = \hat{\mathbf{B}}^{k_n}$, for $n = 1, 2, \ldots$. Let $\Delta\bar{\hat{\mathbf{B}}}$ be the matrix of row sums of $\bar{\hat{\mathbf{B}}}$ and q be the cofactors of the first column of $\Delta\bar{\hat{\mathbf{B}}} - \bar{\hat{\mathbf{B}}}$. The sequences $\{q^{\Gamma k_n}\}$ and $\{q^{k_n}\}$ converge to q as the sequence $\hat{\mathbf{B}}^{k_n}$ converges to $\bar{\hat{\mathbf{B}}}$ and $q^{\Gamma k_n} = q^{k_n}$, for $n = 1, 2, \ldots$. Let $\bar{w} = \max\{\mathbf{w}^1(\tau), \ldots, \mathbf{w}^l(\tau)\}$. Consider the matrix $\hat{\mathbf{B}}^{\Gamma k_n} - \hat{\mathbf{B}}^{\Gamma k_n} \setminus \bar{b}(\tau\rho)$, for $n = 1, 2, \ldots$. Then, $\hat{\mathbf{b}}^{\Gamma k_n}_{ij} - \hat{\mathbf{b}}^{\Gamma k_n}_{ij} \setminus \bar{b}_{ij}(\tau\rho) = (\frac{1}{n}\hat{\mathbf{b}}^{\Gamma k_n}_{ij}(\tau\rho) - \frac{1}{n}\bar{b}_{ij}(\tau\rho))$, $i, j = 1, \ldots, l$, for $n = 1, 2, \ldots$. But then, the sequence of Euclidean distances $\{\|\hat{\mathbf{B}}^{\Gamma k_n} - \hat{\mathbf{B}}^{\Gamma k_n} \setminus \bar{b}(\tau\rho)\|\}$ converges to 0 as $|\frac{1}{n}\hat{\mathbf{b}}^{\Gamma k_n}_{ij}(\tau\rho) - \frac{1}{n}\bar{b}_{ij}(\tau\rho)| = \frac{1}{n}|\hat{\mathbf{b}}^{\Gamma k_n}_{ij}(\tau\rho) - \bar{b}_{ij}(\tau\rho)| \leq \frac{1}{n}\bar{w}$, $i, j = 1, \ldots, l$, $n = 1, 2, \ldots$. The sequence $\{\hat{\mathbf{B}}^{\Gamma k_n} \setminus \bar{b}(\tau\rho)\}$ converges to $\bar{\hat{\mathbf{B}}}$ as, by the triangle inequality, $\|\hat{\mathbf{B}}^{\Gamma k_n} \setminus \bar{b}(\tau\rho)\} - \bar{\hat{\mathbf{B}}}\| \leq \|\hat{\mathbf{B}}^{\Gamma k_n} - \hat{\mathbf{B}}^{\Gamma k_n} \setminus \bar{b}(\tau\rho)\| + \|\hat{\mathbf{B}}^{\Gamma k_n} - \bar{\hat{\mathbf{B}}}\| = \|\hat{\mathbf{B}}^{\Gamma k_n} - \hat{\mathbf{B}}^{\Gamma k_n} \setminus \bar{b}(\tau\rho)\| + \|\hat{\mathbf{B}}^{k_n} - \bar{\hat{\mathbf{B}}}\|$, for $n = 1, 2, \ldots$, and the sequences $\{\|\hat{\mathbf{B}}^{\Gamma k_n} - \hat{\mathbf{B}}^{\Gamma k_n} \setminus \bar{b}(\tau\rho)\|\}$ and $\{\|\hat{\mathbf{B}}^{k_n} - \bar{\hat{\mathbf{B}}}\|\}$ converge to 0.

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Then, the sequence $\{q_{\tau\rho}^{\Gamma^{k_n}}\}$ converges to q. $u_{\tau\rho}(\mathbf{x}(\tau\rho, \hat{\mathbf{b}}^{\Gamma^{k_n}}(\tau\rho), p(\hat{\mathbf{b}}^{\Gamma^{k_n}}))) \geq$
$u_{\tau\rho}(\mathbf{x}(\tau\rho, \hat{\mathbf{b}}^{\Gamma^{k_n}} \setminus \bar{b}(\tau\rho), p(\hat{\mathbf{b}}^{\Gamma^{k_n}} \setminus \bar{b}(\tau\rho))))$ as $\hat{\mathbf{b}}^{\Gamma^{k_n}}$ is a Cobb-Douglas-atom-type-symmetric Cournot-Nash equilibrium of Γ^{k_n}, for $n = 1, 2, \ldots$. Let
$\hat{\mathbf{b}}^{k_n} \setminus \bar{b}(\tau)$ be a strategy selection obtained by replacing $\hat{\mathbf{b}}^{k_n}(\tau)$ in $\hat{\mathbf{b}}^{k_n}$ with $\bar{b}$,
for $n = 1, 2, \ldots$. Then, $u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}}^{k_n}(\tau), q^{\Gamma^{k_n}})) \geq u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}}^{k_n} \setminus \bar{b}(\tau), q_{\tau\rho}^{\Gamma^{k_n}}))$ as
$\hat{\mathbf{b}}^{k_n}(\tau) = \hat{\mathbf{b}}^{\Gamma^{k_n}}(\tau\rho)$, $p(\hat{\mathbf{b}}^{\Gamma^{k_n}}) = \beta_{k_n} q^{\Gamma^{k_n}}$, with $\beta_{k_n} > 0$, by Lemma 2 in Sahi
and Yao (1989), $\hat{\mathbf{b}}^{\Gamma^{k_n}} \setminus \bar{b}(\tau\rho) = \hat{\mathbf{b}}^{k_n} \setminus \bar{b}(\tau)$, and $p(\hat{\mathbf{b}}^{\Gamma^{k_n}} \setminus \bar{b}(\tau\rho)) = \delta_{k_n} q_{\tau\rho}^{\Gamma^{k_n}}$, with
$\delta_{k_n} > 0$, by Lemma 2 in Sahi and Yao (1989), for $n = 1, 2, \ldots$. But then,
$u_\tau(\hat{\mathbf{x}}(\tau)) \geq u_\tau(\bar{x})$, by Assumption 2, as the sequence $\{\hat{\mathbf{b}}^{k_n}(\tau)\}$ converges to
$\hat{\mathbf{b}}(\tau)$, the sequence $\{q^{\Gamma^{k_n}}\}$ converges to q, the sequence $\{q_{\tau\rho}^{\Gamma^{k_n}}\}$ converges to
q, and $\hat{p} = \theta q$, with $\theta > 0$, by Lemma 2 in Sahi and Yao (1989), a contradiction. Consider now the case where $\tau \in T_0$. We have that $\hat{\mathbf{x}}(\tau) \in \mathbf{X}_{\hat{p}}$,
$\bar{x} \in \mathbf{\Delta}_{\hat{p}}(\tau)$, and $u_\tau(\bar{x}) > u_\tau(\hat{\mathbf{x}}(\tau))$, a contradiction
Hence, the pair $(\hat{p}, \hat{\mathbf{x}})$ is a Walras equilibrium of $\mathcal{E}$. ∎

Given the equivalence result proved in Theorem 2, it is immediate to verify that also a Cobb-Douglas-Cournot-Walras equilibrium of $\mathcal{E}$ converges, in the sense of the statement of Theorem 4, to a Walras equilibrium of $\mathcal{E}$.

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